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Diffraction of sea waves by a slender body. Part 1. The shallow-water limit

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A uniformly valid theory (for all wavelengths and angle of incidence) is derived, correct to second order in the slenderness parameter. The leading-order contribution is the unified theory, derived by Newman (1978) and Sclavounous (1982), and it is shown here that for an important class of bodies – those with an 'almost uniform' cross-section – the leading-order correction is of higher order than second, and can be disregarded.

In part 1 of this work the shallow-water limit is considered, and some numerical experiments show good agreement between the observed error and the predicted one. In part 2 (to follow) the theory is extended to the arbitrary-water-depth case, with similar results and accuracy. In both cases it is assumed that the mean forward velocity is zero.

1. Introduction

Slender-body theory has been used in several branches of Applied Mechanics, most notably in aerodynamics. Its application to the diffraction of sea waves by a slender body has been impaired, however, by two difficulties. The first is the fact that there is now an extra lengthscale, the wavelength. The second is that the cross-section problem, essential for the formulation of this theory, was thought to be singular in head-sea incidence. These made it difficult to derive a single theory, uniformly valid for arbitrary wavelength and angle of incidence.

This unified theory was finally derived by Newman and his collaborators. Two landmarks are the works by Newman himself (1978) and Sclavounous (1982), who removed the singularity in head-seas. In spite of this, some practical and theoretical questions do remain.

In fact, their unified theory is valid for an infinite water depth and contains only the leading-order contribution in the small slenderness parameter ϵ. These restrictions can be severe for applications in the offshore industry. Floating structures are usually moored in moderately deep waters and the assumption of infinite water depth can be questionable. Also these structures are not, as a rule, too slender. Barges and semi-submersible platforms, for example, both have a slenderness parameter of order $\epsilon \approx 0.20$ and the leading-order term has then an apparent error of order 20%, which can be unacceptable.

It seems necessary then to develop a unified theory, valid for arbitrary water depth, and correct to second order in the slenderness parameter. This is the purpose of the present paper.

The derivation of such higher-order theory, however, brings out a technical, although important, question: how to measure, in a mathematically consistent way, the error of the asymptotic theory. This question can be properly answered only if we choose some metric to gauge the distance between two functions, but this choice, in general, is never made explicit. Although vagueness on the specific choice of the metric is of no harm when deriving just the leading-order term, it can cause confusion if the purpose is to derive a theory with an error factor of the form $[1+O(\epsilon^2)]$.

Several reasons, to become clear throughout this work, have pushed us to use the metric of generalized functions.† The inner consistency of the theory so obtained makes clear the adequacy of this choice and with it we could formalize some apparently plausible results. Among them one has an important practical application: if a body has a uniform cross-section the leading-order term already has an error factor of the form $[1+O(\epsilon^2)]$. This result motivated us to define a more general class of geometries for which the same sort of behaviour can be shown. We called it the class of bodies with an 'almost uniform' cross-section, and a precise mathematical definition will be given later in this paper. It will become evident then that the slender bodies used in the offshore industry all have an 'almost uniform' cross-section and this is a fortunate circumstance: for these bodies we can use the simpler leading-order theory, keeping the accuracy within the acceptable 4 % margin of error ($\epsilon \approx 0.20$).

To best grasp the subtle aspects of a theory, it is not uncommon to present it first in a simpler context. This is why we have addressed our attention here to the shallow-water limit, where the geometry is relatively simple. The same results, however, can be extended to water of arbitrary depth, as is shown in part 2.

In §2 of the present paper we derive, in a brief way, the boundary-value problem that describes the diffraction of sea waves in shallow water. We also derive there some pertinent results for the related cross-section problem.

In §3 we indicate the basic ideas and results of what we have called the classical approximations, namely, the Froude–Krilov approximation in the long-wave regime, the strip theory in short waves and the parabolic approximation in head-seas. The existence of these approximations in complementary ranges of application suggests the need for a single theory, and guides us to obtain it.

In §§4 and 5 we derive the inner and outer solutions, respectively. The development, so far, is rather standard and follows the ideas of the already classical method of matched asymptotic expansions. The real problem comes when we intend to obtain the inner expansion of the outer solution. To derive it in a mathematically consistent way we analyse, in §6, the Fourier transform in a certain class-B of functions. In this section and in the Appendix we introduce the whole mathematical apparatus of the present work.

In §7 we derive the inner expansion of the outer solution, correct to an error of order ϵ^2 compared with 1. In §8 we complete the theory, matching the inner and outer solutions, and we analyse also the real effect of the second-order term for a body with a uniform cross-section. We introduce then, in a precise mathematical way, the class of bodies with an 'almost uniform' cross-section, and prove that the error factor of the leading term is of the form $[1+O(\epsilon^2)]$ for this class of geometries. In §9 we present some numerical experiments that confirm the main results of the theory.

2. The shallow-water equations

In this paper a raft resting on the free surface, symmetric with respect to the longitudinal axis, will be analysed. The geometric definitions are given in figure 1.

The surface of the body will be denoted by S. The water is said to be 'shallow' if $K_0 h \ll 1$, where K_0 is the wavenumber and h is the water depth. The total potential

† It would be correct to say 'to measure the error in the space of the generalized functions' since there is no metric strict sense defined in this space.

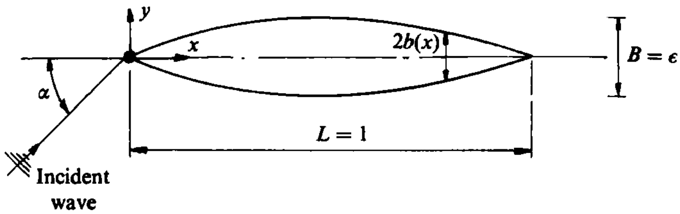

FIGURE 1. Geometric definitions ($\epsilon = B/L$).

$\hat{\Phi}_T$ is the sum of the incident wave $\hat{\Phi}_I$ and the scattered potential $\hat{\Phi}$. In the horizontal plane $z = 0$ it satisfies the boundary condition $\partial\hat{\Phi}_T/\partial z = (\omega^2/g)\,\hat{\Phi}_T$ if $(x,y) \notin \mathrm{S}$, and $\partial\hat{\Phi}_T/\partial z = 0$ if $(x,y) \in \mathrm{S}$. In the first expression ω is the wave frequency, and it is related to the wavenumber K_0 by the dispersion relation $\omega^2 h/g = (K_0 h)^2\,[1+O(K_0 h)^2]$. Integrating vertically the potential equation one obtains, with an error of the form $[1+O(K_0 h)^2]$, the set of equations (see Yue & Mei 1981)

(i) field equation,

$$\overline{\nabla}^2\hat{\Phi}_T + K_0^2\,\hat{\Phi}_T = 0 \quad \text{for } (x,y) \notin \mathrm{S} \quad \left(\overline{\nabla}^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

(ii) boundary condition on body's surfaces,

$$\overline{\nabla}^2\hat{\Phi}_T = 0 \quad \text{for } (x,y) \in \mathrm{S},$$

(iii) radiation condition,

$$(\hat{\Phi}_T - \hat{\Phi}_I) \sim CH_0^{(1)}(K_0 r) \quad \text{as } r = (x^2+y^2)^{\frac{1}{2}} \to \infty.$$

This last condition is the standard outgoing condition for the scattered wave. We implicitly assumed the time factor $\exp(-\mathrm{i}\omega t)$, and the function $H_0^{(1)}(\,.\,)$ is the Hankel function of the first kind.

The incident wave is here given by

$$\left.\begin{aligned}\hat{\Phi}_I(x,y) &= \phi_I(y)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha},\\ \phi_I(y) &= \mathrm{e}^{\mathrm{i}K_0 y\sin\alpha}.\end{aligned}\right\} \tag{2.1}$$

Consistent with (2.1) we define also

$$\left.\begin{aligned}\hat{\Phi}_T(x,y) &= \Phi_T(x,y)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha},\\ \hat{\Phi}(x,y) &= \Phi(x,y)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha}.\end{aligned}\right\} \tag{2.2}$$

Placing (2.1) and (2.2) into the set of equations above we obtain:

(i) field equation,

$$\frac{\partial^2\Phi}{\partial y^2} + (K_0\sin\alpha)^2\Phi + 2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi}{\partial x} + \frac{\partial^2\Phi}{\partial x^2} = 0 \quad \text{for } (x,y) \notin \mathrm{S}, \tag{2.3}$$

(ii) boundary condition,

$$\frac{\partial^2\Phi}{\partial y^2} - (K_0\cos\alpha)^2\Phi + 2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi}{\partial x} + \frac{\partial^2\Phi}{\partial x^2} = K_0^2\,\phi_I(y) \quad \text{for } (x,y) \in \mathrm{S}, \tag{2.4}$$

(iii) radiation condition,

$$\Phi(x,y)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha} \sim CH_0^{(1)}(K_0 r) \quad \text{as } r \to \infty. \tag{2.5}$$

We should notice that the scattered potential is now being excited by a term that does not change with x. For an infinitely long body $(1/\epsilon = \infty)$ we expect that both

$\partial\Phi/\partial x$ and $\partial^2\Phi/\partial x^2$ should be zero and, in this case, the scattered wave should be determined by the cross-section problem. This is essential for the slender-body theory.

The potential $\Phi(x,y)$ can be expressed with the help of the Green function

$$G(x-\xi;\,y-\eta) = -\tfrac{1}{4}\mathrm{i}H_0^{(1)}(K_0((x-\xi)^2+(y-\eta)^2)^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}K_0(x-\xi)\cos\alpha}, \tag{2.6}$$

and it can be written as

$$\Phi(x,y) = \int_0^L \mathrm{d}\xi \int_{-b(\xi)}^{b(\xi)} \mathrm{d}\eta\,\sigma(\xi,\eta)\,G(x-\xi;\,y-\eta). \tag{2.7}$$

The source density $\sigma(\xi,\eta)$ is the solution of the integral equation defined below,

$$\left.\begin{aligned} \sigma(x,y) &= K_0^2\,\Phi_{\mathrm{T}}(x,y) \quad \text{for } (x,y)\in \mathrm{S},\\ \sigma(x,y)-K_0^2\int_0^L \mathrm{d}\xi \int_{-b(\xi)}^{b(\xi)} \mathrm{d}\eta\,\sigma(\xi,\eta)\,G(x-\xi;\,y-\eta) &= K_0^2\,\phi_{\mathrm{I}}(y) \quad \text{for } (x,y)\notin \mathrm{S}. \end{aligned}\right\} \tag{2.8}$$

This equation will be used later, in §9, when the results of the slender-body theory will be contrasted with the ones obtained from the full linear theory.

2.1. *The cross-section problem*

For an infinitely long uniform body, $\partial\Phi/\partial x = \partial^2\Phi/\partial x^2 = 0$,† and from (2.3), (2.4) and (2.5) we obtain,

(i) field equation,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}+(K_0\sin\alpha)^2\phi = 0 \quad (|y|>b), \tag{2.9}$$

(ii) boundary condition,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}-(K_0\cos\alpha)^2\phi = K_0^2\,\phi_{\mathrm{I}}(y) \quad (|y|<b), \tag{2.10}$$

(iii) radiation condition,

$$\phi(y) \sim \begin{Bmatrix} T \\ R \end{Bmatrix} \mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha} \quad \text{when } y\to\pm\infty. \tag{2.11}$$

The above set of equations defines the cross-section problem. In (2.11), R and $T+1$ are the reflection and transmission coefficients, respectively.

The solution of (2.9), (2.10), (2.11) is given by

$$\left.\begin{aligned} \phi^{\pm}(y) &= [A_0^{\pm}+H(\alpha)\,K_0\,U(|y|-b)]\,\mathrm{e}^{\mathrm{i}K_0\sin\alpha(|y|-b)} \quad (y \gtrless \pm b),\\ \phi(y) &= -\phi_{\mathrm{I}}(y)+E\cosh(K_0 y\cos\alpha)+F\frac{\sin(K_0 y\cos\alpha)}{\sinh(K_0 b\cos\alpha)} \quad (|y|\leqslant b), \end{aligned}\right\} \tag{2.12a}$$

where $H(\alpha)$ is the discontinuous function

$$H(\alpha) = \begin{cases} 0 & \text{if } 0<\alpha\leqslant\frac{1}{2}\pi,\\ 1 & \text{if } \alpha = 0. \end{cases} \tag{2.12b}$$

We have introduced this function together with the parameter U, because of a peculiarity in head-seas ($\alpha = 0$) to be explained later in this section. In what follows the value of U will be supposed known.

† Throughout this work the symbols $\{\phi(y);\,\phi_{\mathrm{T}}(y)\}$ will represent the cross-section solution. They could change with x only due to the variation of the half-beam $b = b(x)$. This dependence, however, will in general be omitted.

The four unknowns of (2.12*a*), $\{A_0^+, A_0^-, E, F\}$, can be determined by enforcing continuity of pressure and flux along the line $y = \pm b$. In this way we can easily check that the coefficients $A_0^\pm$ are solutions of the system

$$\left.\begin{aligned} \boldsymbol{A}\begin{Bmatrix} A_0^+ \\ A_0^- \end{Bmatrix} &= \begin{Bmatrix} V_0^+ \\ V_0^- \end{Bmatrix} + H(\alpha)\, U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \\ \boldsymbol{A} = \text{scattering matrix} &= \begin{bmatrix} a_1 - \mathrm{i}\sin\alpha & a^2 \\ a_2 & a_1 - \mathrm{i}\sin\alpha \end{bmatrix}, \end{aligned}\right\} \tag{2.13a}$$

where

$$\left.\begin{aligned} a_1 &= \tfrac{1}{2}\cos\alpha[\operatorname{cotanh}(K_0 b\cos\alpha) + \tanh(K_0 b\cos\alpha)], \\ a_2 &= \tfrac{1}{2}\cos\alpha[-\operatorname{cotanh}(K_0 b\cos\alpha) + \tanh(K_0 b\cos\alpha)], \end{aligned}\right\} \tag{2.13b}$$

$$\begin{aligned} V_0^\pm = -\cos\alpha[&\tanh(K_0 b\cos\alpha)\cos(K_0 b\sin\alpha) \\ &\mp \mathrm{i}\operatorname{cotanh}(K_0 b\cos\alpha)\sin(K_0 b\sin\alpha)] \mp \mathrm{i}\sin\alpha\, \mathrm{e}^{\mp \mathrm{i}K_0 b\sin\alpha}. \end{aligned}$$

In head-seas, the scattering matrix is real and has two eigenvalues

$$\left.\begin{aligned} A_1 &= a_1 + a_2 = \tanh K_0 b, \\ A_2 &= a_1 - a_2 = \operatorname{cotanh} K_0 b. \end{aligned}\right\} \tag{2.14}$$

They will be needed later in this section. Before we give the explicit expressions for $\{E; F\}$ it is convenient to introduce some definitions. Let

$$\left.\begin{aligned} R_{\mathrm{S}}(\alpha) &= \tfrac{1}{2}(T+R), \\ R_{\mathrm{A}}(\alpha) &= \tfrac{1}{2}(T-R), \end{aligned}\right\} \tag{2.15a}$$

where, from (2.11) and (2.12*a*),

$$\begin{Bmatrix} T \\ R \end{Bmatrix} = \begin{Bmatrix} A_0^+ \\ A_0^- \end{Bmatrix} \mathrm{e}^{-\mathrm{i}K_0 b\sin\alpha} - H(\alpha)\, K_0 b U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{2.15b}$$

Adding the incident wave to (2.12*a*), we obtain the total wave in the cross-section. In the region $|y| \geqslant b$ its expression is given by:

$$\left.\begin{aligned} \phi_{\mathrm{T}}(y) &= \phi_{\mathrm{T,S}}(y) + (\operatorname{sign} y)\,\phi_{\mathrm{T,A}}(y), \\ \phi_{\mathrm{T,S}}(y) &= \cos(K_0|y|\sin\alpha) + R_{\mathrm{S}}(\alpha)\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha} + H(\alpha)\,K_0\, U|y|, \\ \phi_{\mathrm{T,A}}(y) &= \mathrm{i}\sin(K_0|y|\sin\alpha) + R_{\mathrm{A}}(\alpha)\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}. \end{aligned}\right\} \tag{2.16}$$

In (2.16), $\{\phi_{\mathrm{T,S}}; \phi_{\mathrm{T,A}}\}$ are the symmetric and antisymmetric parts of ϕ_{T}, respectively, and sign $y = \pm 1$ if $y \gtrless 0$.

It is not difficult to check now that

$$\left.\begin{aligned} E &= \frac{1}{\cosh(K_0 b\cos\alpha)}[R_{\mathrm{S}}(\alpha)\,\mathrm{e}^{\mathrm{i}K_0 b\sin\alpha} + \cos(K_0 b\sin\alpha)] \\ F &= R_{\mathrm{A}}(\alpha)\,\mathrm{e}^{\mathrm{i}K_0 b\sin\alpha} - \mathrm{i}\sin(K_0 b\sin\alpha). \end{aligned}\right\} \tag{2.17}$$

We close this section with some results that will be needed later. They can be easily derived from the expressions given above.

(*a*) $K_0 B \ll 1$

Then

$$\left.\begin{matrix} R_S(\alpha) \\ R_A(\alpha) \\ \phi(y;\alpha) \end{matrix}\right\} \sim O(K_0 B) \quad \text{when } K_0 B \to 0, \quad \sin\alpha \sim O(1). \tag{2.18a}$$

Actually a stronger result can be derived for $R_A(\alpha)$. It is

$$R_A(\alpha) \sim O(K_0 B)^2 \quad \text{when } K_0 B \to 0, \quad \sin\alpha \sim O(1). \tag{2.18b}$$

Expression (2.18*b*) is, however, a peculiarity of the shallow-water limit, or for bodies with shallow draught.

(*b*) $0 < \sin\alpha \ll 1$

In this case

$$R_S(\alpha) \sim \left[-1 - \mathrm{i}\sin\alpha\left(\frac{1}{\Lambda_1} - K_0 b\right)\right](1 + O(\sin^2\alpha)). \tag{2.19a}$$

$$R_A(\alpha) \sim \mathrm{i}\sin\alpha\left(\frac{1}{\Lambda_2} - K_0 b\right)(1 + O(\sin^2\alpha)). \tag{2.19b}$$

when $\sin\alpha \to 0$. In (2.19), Λ_1, Λ_2 are the eigenvalues of the scattering matrix in head-seas; see (2.14).

In this limit we can also verify the important relation (for $|y| \leqslant b$)

$$\phi_{T,S}(y;\alpha) \sim -\frac{\mathrm{i}\sin\alpha}{\Lambda_1}\frac{\cosh K_0 y}{\cosh K_0 b} \quad (|\sin\alpha| \to 0). \tag{2.20}$$

The above expression shows that the total potential, although continuous in α, tends to zero as $\alpha \to 0$. Or, in other words: $\phi_T(y;\alpha = 0) = 0$ if $U = 0$ in (2.12*a*). To avoid this trivial solution we have added the term proportional to U in head-seas.

(*c*) $\alpha = 0$

In this case

$$R_S(\alpha)|_{\alpha=0} = -1 + U\left(\frac{1}{\Lambda_1} - K_0 b\right), \tag{2.21a}$$

$$R_A(\alpha)|_{\alpha=0} = 0, \tag{2.21b}$$

and also

$$\left.\begin{aligned} \phi_T(y;\alpha)|_{\alpha=0} &= \frac{U}{\Lambda_1}\frac{\cosh K_0 y}{\cosh K_0 b} \quad (|y| \leqslant b), \\ \phi_T(y;\alpha)|_{\alpha=0} &= \left[-1 + U\left(\frac{1}{\Lambda_1} - K_0 b\right)\right] + K_0 U|y| \quad (|y| \geqslant b). \end{aligned}\right\} \tag{2.22}$$

A close look at the expressions (2.19), (2.21*a*) and (2.20), (2.22) indicates that the place of $-\mathrm{i}\sin\alpha$ when $\alpha \neq 0$ is taken by U in the limit $\alpha = 0$. This is certainly not accidental but it is a reflection of the fact that the cross-section problem is well-behaved and regular in the limit $\alpha = 0$. These results are essential for the unified theory.

3. Classical theories

It is usual, in ocean engineering, to say that a wave is long if its wavelength is of the same order of magnitude as the ship's length ($K_0 L \sim O(1)$); if it is of the order of the ship's beam, the wave is said to be short ($K_0 B \sim O(1)$). This nomenclature will be followed in this work, although the theory to be developed is also valid for very long and very short waves, $K_0 L \ll 1$ and $K_0 B \gg 1$, respectively.

The existence of an extra lengthscale – the wavelength – and the peculiar behaviour of the cross-section problem in head-seas, as discussed in §3, make it difficult to derive a single asymptotic theory, valid irrespective of the wavelength and angle of incidence.

The best way to understand the difficulty is to present three asymptotic theories, valid in complementary ranges of application. We have named them 'classical theories' although one, the parabolic approximation, has only recently been introduced in the field of sea waves, by Mei & Tuck (1980). All of them, however, share a common property: they are relatively simple and represent the leading-order contribution.

3.1. *Froude–Krilov approximation (long waves)*

For long waves, $K_0 B \sim O(\epsilon)$ and the scattered potential is of order ϵ, see (2.18*a*). Then,

$$\Phi_{\mathrm{T}}(x,y) = \phi_{\mathrm{I}}(y) + O(\epsilon). \tag{3.1}$$

This is the Froude–Krilov approximation.

3.2. *Strip theory (short waves; $\sin\alpha \sim O(1)$)*

In this case both the wavelength and the beam B are small compared with the length L. In the first instance the body can be considered as if it had an infinite length, and the diffraction problem then coincides with the cross-section problem. So,

$$\Phi_{\mathrm{T}}(x,y) = \phi_{\mathrm{T}}(y)(1+O(\epsilon)) = [\phi_{\mathrm{I}}(y)+\phi(y)](1+O(\epsilon)). \tag{3.2}$$

This is the strip theory, and the error factor shown will be confirmed later. It is important to observe also that strip theory, although deduced for short waves, can also be used in the long-wave regime, since then $\phi(y) \sim O(\epsilon)$; see (2.18*a*) and (3.1).

3.3. *Parabolic approximation (short waves; $\alpha = 0$)*

If the wave is short and $\alpha = 0$, the diffraction problem is, in the first instance, similar to a head-sea incident on a semi-infinite cylinder. From this idea it is possible to approximate the field equation (2.3) by a parabolic equation, in a region relatively far from the body. Details can be found in Mei & Tuck (1980), but the expansion of this solution when $|y| \to 0$ is given by

$$\left.\begin{aligned} \Phi(x,y;0) &\sim -\tfrac{1}{2}\mathrm{i}J(x) + \tfrac{1}{2}\mathrm{i}|y|\,Q(x;0), \\ J(x) &= \frac{1-\mathrm{i}}{2(\pi K_0)^{\frac{1}{2}}}\int_0^x \mathrm{d}\xi\,\frac{Q(\xi;0)}{(x-\xi)^{\frac{1}{2}}}. \end{aligned}\right\} \tag{3.3a}$$

Near the body the cross-section equation holds, and the solution is given by (2.22) with $|y| \geqslant b(x)$. By matching we determine $U(x) = Q(x;0)/2K_0$ where $Q(x)$ is the solution of Abel's integral equation

$$\left.\begin{aligned} 1-\frac{1+\mathrm{i}}{4(\pi K_0)^{\frac{1}{2}}}\int_0^x \frac{\mathrm{d}\xi Q(\xi;0)}{(x-\xi)^{\frac{1}{2}}} &= \sigma Q(x;0), \\ \sigma = \frac{1}{2K_0 A_1}(1-K_0 b A_1) \quad (A_1 &= \tanh K_0 b). \end{aligned}\right\} \tag{3.3b}$$

This is the parabolic approximation. A similar result was first obtained by Faltinsen (1971) and further perfected by Maruo & Sasaki (1974). The outer solution here, however, is much easier than the one derived by Faltinsen and, in line with the simplicity of the other classical theories, we have chosen the parabolic approximation to represent the diffraction in head-sea.

In long waves the solution of (3.3b) is of the form $Q(x) \sim 2K_0 A_1(1+O(A_1))$, and so $\Phi_{\mathrm{T}}(x,y) = 1+O(\epsilon)$; see (3.3$a$). So the parabolic approximation is equivalent to Froude–Krilov in long waves and can also be used in this limit.

4. The inner solution

The method of matched asymptotic expansions distinguishes two regions: one close to the body, called the 'inner region', where $|y|/B \leqslant O(1)$, and the other far from it, called the 'outer region', where $|y|/L \geqslant O(1)$. In the inner region one feels the body's boundary condition but misses the radiation condition. In the outer region the opposite holds. The indeterminacy of each one is resolved by matching the inner and outer solutions in an 'overlap' region.

In this section the inner solution will be worked out. In this case the radiation condition is missing and we must look only to (2.3) and (2.4).

These equations can, however, be simplified. In fact, the lengthscale in the x-direction is L, the ship's length. So $\partial\Phi/\partial x \sim O(\Phi/L)$. In the inner region the lengthscale in the y-direction is B, the ship's beam. So $\partial\Phi/\partial y \sim O(\Phi/B)$. The term $\partial^2\Phi/\partial x^2$ is then of relative order ϵ^2 and can be disregarded. We obtain, in this way:

(i) field equation,

$$\frac{\partial^2\Phi_{\mathrm{i}}}{\partial y^2}+(K_0\sin\alpha)^2\Phi_{\mathrm{i}} = -2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi_{\mathrm{i}}}{\partial x} \quad (|y| \geqslant b(x)); \tag{4.1}$$

(ii) boundary condition,

$$\frac{\partial^2\Phi_{\mathrm{i}}}{\partial y^2}-(K_0\cos\alpha)^2\Phi_{\mathrm{i}} = K_0^2\phi_{\mathrm{I}}(y)-2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi_{\mathrm{i}}}{\partial x} \quad (|y| \leqslant b(x)).$$

The suffix i is to indicate that this is the inner solution. The term $2\mathrm{i}K_0\cos\alpha\,\partial\Phi_{\mathrm{i}}/\partial x$ is of relative order ϵ in short waves and must be kept, since we want to derive a theory correct with an error of the order ϵ^2.

It is convenient then to write:

$$\Phi_{\mathrm{i}}(x,y) = \Phi_{\mathrm{i}}^{(1)}(x,y)+\Phi_{\mathrm{i}}^{(2)}(x,y), \quad \frac{\Phi_{\mathrm{i}}^{(2)}}{\Phi_{\mathrm{i}}^{(1)}} \sim O(\epsilon). \tag{4.2}$$

Placing (4.2) into (4.1) and separating terms of like orders we obtain:

(*a*) *leading order* ($O(1)$)

$$\left.\begin{aligned}\frac{\partial^2 \Phi_{\mathrm{i}}^{(1)}}{\partial y^2}+(K_0 \sin\alpha)^2 \Phi_{\mathrm{i}}^{(1)} &= 0 \quad (|y| \geqslant b(x)),\\ \frac{\partial^2 \Phi_{\mathrm{i}}^{(1)}}{\partial y^2}-(K_0 \cos\alpha)^2 \Phi_{\mathrm{i}}^{(1)} &= K_0^2\, \phi_{\mathrm{I}}(y) \quad (|y| \leqslant b(x)).\end{aligned}\right\} \tag{4.3}$$

(*b*) *second order* ($O(\epsilon)$)

$$\left.\begin{aligned}\frac{\partial^2 \Phi_{\mathrm{i}}^{(2)}}{\partial y^2}+(K_0 \sin\alpha)^2 \Phi_{\mathrm{i}}^{(2)} &= -2\mathrm{i}K_0 \cos\alpha \frac{\partial \Phi_{\mathrm{i}}^{(1)}}{\partial x} \quad (|y| \geqslant b(x)),\\ \frac{\partial^2 \Phi_{\mathrm{i}}^{(2)}}{\partial y^2}-(K_0 \cos\alpha)^2 \Phi_{\mathrm{i}}^{(2)} &= -2\mathrm{i}K_0 \cos\alpha \frac{\partial \Phi_{\mathrm{i}}^{(1)}}{\partial x} \quad (|y| \leqslant b(x)).\end{aligned}\right\} \tag{4.4}$$

Next, we shall analyse both sets of equations.

4.1. *The leading-order inner solution*

The most general solution of (4.3) is the sum† of a particular solution with the homogeneous one. A particular solution can be easily found, it is just the incident wave with negative sign.

Equation (4.3) has two independent homogeneous solutions and a convenient way to express them has been proposed by Newman (1978). In fact, he observed that the cross-section total wave $\phi_{\mathrm{T}}(y)$ is a homogeneous solution of (4.3). So its symmetric and antisymmetric parts are two linearly independent homogeneous solutions of (4.3) and then the most general expression for $\Phi_{\mathrm{i}}^{(1)}(x,y)$ is given by (see (2.16)),

$$\begin{aligned}\Phi_{\mathrm{i}}^{(1)}(x,y) = {}& [S_1(x)\,\phi_{\mathrm{T,S}}(y)-\cos(K_0|y|\sin\alpha)]\\ &+\operatorname{sign} y[A_1(x)\,\phi_{\mathrm{T,A}}(y)-\mathrm{i}\sin(K_0|y|\sin\alpha)].\end{aligned} \tag{4.5}$$

The two arbitrary 'constants' $\{S_1(x); A_1(x)\}$ must, in general, depend on x and they will be determined by matching with the outer solution. In the strip theory the radiation condition (2.11) applies and the values of $\{S_1(x); A_1(x)\}$ can be determined. They are obviously $S_1(x) = A_1(x) = 1$.

4.2. *The second-order inner solution*

The only difference here is the determination of a particular solution. Let $\overline{\Phi}_{\mathrm{P}}(x,y)$ be this function, where

$$\overline{\Phi}_{\mathrm{P}}(x,y) = \overline{\Phi}_{\mathrm{P,S}}(x,y)+(\operatorname{sign} y)\,\overline{\Phi}_{\mathrm{P,A}}(x,y). \tag{4.6}$$

An explicit expression for $\overline{\Phi}_{\mathrm{P}}(x,y)$ can be worked out, but its expression will be omitted here. Once this function is known, we can write

$$\Phi_{\mathrm{i}}^{(2)}(x,y) = [S_2(x)\,\phi_{\mathrm{T,S}}(y)+\overline{\Phi}_{\mathrm{P,S}}(x,y)]+(\operatorname{sign} y)\,[A_2(x)\,\phi_{\mathrm{T,A}}(y)+\overline{\Phi}_{\mathrm{P,A}}(x,y)]. \tag{4.7}$$

The two 'constants' $\{S_2(x); A_2(x)\}$ can, again, be determined by matching with the second-order outer solution.

† For 'homogeneous solution' we obviously understand a non-trivial solution of the homogeneous system associated with (4.3) or (4.4).

The potential $\Phi_1^{(2)}(x,y)$ is being forced by the term $-2\mathrm{i}K_0\cos\alpha\,\partial\Phi_1^{(1)}/\partial x$. Since $B=\epsilon$, $L=1$ then, using the scaling† $\partial^2\Phi/\partial y^2 \sim O(\Phi/\epsilon^2)$, we obtain

$$\Phi_1^{(2)}(x,y) \sim O\left(\epsilon K_0 B\cos\alpha\frac{\partial\Phi_1^{(1)}}{\partial x}\right). \tag{4.8a}$$

The above expression is important for the following reason: using the strip-theory approximation in (4.5) we get

$$\Phi_1^{(1)}(x,y) = [\phi_{\mathrm{T}}(y;x)-\phi_{\mathrm{I}}(y)](1+O(\epsilon)), \tag{4.8b}$$

where $\phi_{\mathrm{T}}(y;x)$ depends on x only owing to the variation of the cross-section in the longitudinal direction. In general $\partial\phi_{\mathrm{T}}/\partial x \sim O(1)$ and, from (4.8*a*), $\Phi_1^{(2)} \sim O(\epsilon)$ and must be computed. For a uniform cross-section, however, $\partial\phi_{\mathrm{T}}/\partial x = 0$ and, from (4.8*b*), we obtain $\partial\Phi_1^{(1)}/\partial x \sim O(\epsilon)$. In this case $\Phi_1^{(2)} \sim O(\epsilon^2)$ and it need not be computed. Or, in other words: the above reasoning seems to indicate that the leading-order term already has an error factor $[1+O(\epsilon^2)]$, when the cross-section is uniform. One of the outcomes of the present paper is just the formalization of this seemingly plausible result.

5. Outer solution

In the outer region the body's boundary condition is missed. So the 'outer solution' is a general solution of (2.3) and (2.5). But, since it must be matched with the inner solution, it should contain four arbitrary functions: $\{Q_1(x);Q_2(x)\}$, associated with the symmetric coefficients $\{S_1(x);S_2(x)\}$, and $\{M_1(x);M_2(x)\}$, associated with $\{A_1(x);A_2(x)\}$.

For an observer in the outer region the slender body looks like a line emitting waves. It is natural then to write (see (2.6))

$$\left.\begin{aligned}\Phi_0(x,y) &= \int_0^L \mathrm{d}\xi\left[Q(\xi)+\frac{1}{K_0}M(\xi)\frac{\partial}{\partial y}\right]G(x-\xi;y),\\ Q(\xi) &= Q_1(\xi)+Q_2(\xi),\\ M(\xi) &= M_1(\xi)+M_2(\xi),\\ \left\{\frac{Q_2}{Q_1};\frac{M_2}{M_1}\right\} &\sim O(\epsilon).\end{aligned}\right\} \tag{5.1}$$

In fact (5.1) is a solution of (2.3) and (2.5) and contains just the four functions needed for the matching. The suffix o is to indicate that this is the outer solution.

The next step would be to match the inner and outer solution in an 'overlap region', where $B \ll |y| \ll L$. For this we need the 'inner expansion of the outer solution' – that is, the asymptotic behaviour of $\Phi_0(x,y)$ when $|y|\to 0$ – and also the 'outer expansion of the inner solution', or the behaviour of $\Phi_{\mathrm{i}}(x,y)$ when $|y|\to 0$.

The 'outer expansion of the inner solution' is always trivial. In this case it is given directly by (2.16) and (4.5) and the 'overlap region' coincides with the 'inner region'. As we are going to see in part 2 of this work, this 'outer expansion' is equally easy in the arbitrary-water-depth case.

The real problem is to derive the 'inner expansion of the outer solution'. An

† For long waves $\partial^2\Phi/\partial x^2$ is of the same order as $-2\mathrm{i}K_0\cos\alpha\,\partial\Phi/\partial x$. In this case, also, the order of magnitude of $\Phi^{(2)}$ is given by (4.8*a*).

ingenious way to overcome this difficulty is indicated in Newman (1978) and it is based on the following observation: the line integral that defines the 'outer solution' has the structure of a convolution integral. It seems natural, then, to take the Fourier transform of (5.1) with respect to x, to determine its limit when $|y| \to 0$ and then to take the inverse Fourier transform to get the 'inner expansion of the outer solution'.

Both Newman (1978) and Sclavounous (1982) were concerned just with the leading-order term so they didn't have to worry too much about the actual behaviour of the Fourier transforms of the densities $\{Q(x); M(x)\}$. Here, however, we want to go a step further and it seems unlikely that we could have much success if such behaviour were not well understood. To this issue we dedicate the next section.

6. The Fourier transform in the class B of functions

In order to introduce a convenient class of functions where the densities $\{Q(x); M(x)\}$ can be located we follow an alternative route and deduce (5.1) directly from the representation of the scattered wave in the full linear theory; see (2.7).

This is quite easy in the long-wave regime. In fact, the Green function in (2.7) is analytic in the outer region ($|y|/L \geqslant O(1)$) and for long waves ($K_0 B \sim O(\epsilon)$) we can write

$$G(x-\xi; y-\eta) = \left[G(x-\xi; y) - \frac{1}{K_0}\frac{\partial G}{\partial y}(x-\xi; y)(K_0 \eta)\right](1+O(\epsilon^2)),$$

since $|\eta| \leqslant B$. Placing this expansion into (2.7) we obtain

$$\Phi(x, y) = \Phi_0(x, y)(1+O(\epsilon^2)) \tag{6.1a}$$

with

$$\left.\begin{aligned} Q(\xi) &= \int_{-b(\xi)}^{b(\xi)} \sigma(\xi, \eta)\, \mathrm{d}\eta, \\ M(\xi) &= \int_{-b(\xi)}^{b(\xi)} \sigma(\xi, \eta)\, K_0 \eta\, \mathrm{d}\eta. \end{aligned}\right\} \tag{6.1b}$$

Expression (6.1b) gives an indication about the class of functions where the densities are. This class is the same as that containing the function $b(x)$, since the source density $\sigma(\xi, \eta)$, being just the total wave beneath the raft, is a regular function of ξ.

Later in this paper we will confirm, in a more formal way, that the densities have the same singularities as $b(x)$ has. We introduce now the *form function* $g(x)$, where

$$g(x) = \frac{2b(x)}{B}, \tag{6.2a}$$

$$g(0) = \lim_{x \to 0^+} g(x), \quad g(L) = \lim_{x \to L^-} g(x). \tag{6.2b}$$

Obviously $g(x)$ defines completely the geometry of the body and we can easily introduce now a class B of functions that covers all geometries of practical interest. This class will be designated by $B[0; L]$ and it has the following properties:

(i) $g(x) \equiv 0$ if $x < 0$ or $x > L$;

(ii) $g(x)$ is continuous in $0 < x < L$ although $\mathrm{d}g/\mathrm{d}x$ may be discontinuous at $x_0 = 0, x_1, x_2, \ldots, x_e = L$;

(iii) the singular points of $\mathrm{d}g/\mathrm{d}x$ are well separated, in the sense that $|x_{j+1} - x_j| \sim O(1)$ for $j = 0, 1, \ldots, (e-1)$;

(iv) all derivatives of $g(x)$ exist, are continuous and of order 1 if $x_j^+ \leqslant x \leqslant x_{j+1}^-$.

Obviously $g(x)$ can be discontinuous at $x = 0$ and $x = L$, as it should be for a body with blunt ends. An important subclass of $B[0;L]$ is formed by the bodies with pointed ends, defined below:

$$B_{\mathrm{E}}[0;L] = \{g(x) \in B[0;L] \quad \text{such that } g(0) = g(L) = 0\}.$$

The class $B[0;L]$ is sufficiently well behaved and we can derive some sharp results related to its Fourier transform. They are described below and elaborated in the Appendix.

6.1. *Fourier transforms, and the measure of the error*

Let $P(x) \in B[0;L]$ and consider the Fourier-transforms pair

$$\left.\begin{aligned} P^*(K) &= F[P(x)] = \int_{-\infty}^{\infty} P(x)\,\mathrm{e}^{\mathrm{i}Kx}\,\mathrm{d}x, \\ P(x) &= F^{-1}[P^*(K)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} P^*(K)\,\mathrm{e}^{-\mathrm{i}Kx}\,\mathrm{d}K. \end{aligned}\right\} \tag{6.3}$$

As we shall see later in this section, the Fourier transform $G^*(y;K)$ of the Green function changes its behaviour drastically in short waves, when $|K| > O(K_0) \sim O(1/\epsilon)$. It turns out that to determine, in expression (5.1), the limit of $\Phi_0^*(y;K)$ when $|y| \to 0$ we must be restricted to the region $|K| \leqslant O(1/\epsilon)$. In this way the densities $P(x) = \{Q(x); M(x)\}$ are actually represented by the function

$$\overline{P}(x) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} P^*(K)\,\mathrm{e}^{-\mathrm{i}Kx}\,\mathrm{d}K \quad (\lambda \gg 1), \tag{6.4}$$

where, in our case, $\lambda \sim O(1/\epsilon)$.

It is then necessary to check how closely $\overline{P}(x)$ represents $P(x)$. It can be shown (see Appendix) that

$$P(x) \doteq \left[\overline{P}(x) + \sum_{j=0}^{e} \Delta\dot{P}_j\, D_j(x) + P(0)\frac{\mathrm{d}D_0}{\mathrm{d}x}(x) + P(L)\frac{\mathrm{d}D_e}{\mathrm{d}x}(x)\right]\left(1 + O\left(\frac{1}{\lambda^2}\right)\right), \tag{6.5}$$

where

$$\left.\begin{aligned} \Delta\dot{P}_j &= \frac{\mathrm{d}P}{\mathrm{d}x}(x_j^+) - \frac{\mathrm{d}P}{\mathrm{d}x}(x_j^-) \quad (j = 0, 1, \ldots, e), \\ D_j(x) &= -\frac{1}{2\pi\lambda} D(\lambda|x - x_j|), \\ D(z) &= -2\int_z^{\infty} \mathrm{si}\,(t)\,\mathrm{d}t, \\ \mathrm{si}\,(t) &= -\int_t^{\infty} \frac{\sin\xi}{\xi}\,\mathrm{d}\xi. \end{aligned}\right\} \tag{6.6}$$

For a body with pointed ends ($P(0) = P(L) = 0$) we obtain the simpler relation

$$\left.\begin{aligned} P(x) &\doteq \left[\overline{P}(x) + \sum_{j=0}^{e} \Delta\dot{P}_j\, D_j(x)\right]\left(1 + O\left(\frac{1}{\lambda^2}\right)\right), \\ &P(x) \in B_{\mathrm{E}}[0;L]. \end{aligned}\right\} \tag{6.7}$$

If we take $\lambda \sim O(1/\epsilon)$, as in general we will, then (6.7) has the desired error factor. A close look at $D_j(x)$ reveals that this function has a peculiar structure. Indeed, it is of order $1/\lambda^2$ if $|x-x_j| \sim O(1)$ and of order $1/\lambda$ when $|x-x_j| \leqslant O(1/\lambda)$. Or, in short, it is of order $1/\lambda^2$ 'almost everywhere' (in this loose, but not strict, sense). This behaviour can be formalized if we gauge $D_j(x)$ in the space of distribution theory. With this motivation we introduce the following definition:

Definition 1. We say that

$$f(x) = g(x)\,(1+O(\epsilon^\beta)) \quad (\beta > 0), \tag{6.8a}$$

if

$$I(\Psi) = \int_{-\infty}^{\infty} [f(x)-g(x)]\,\Psi(x)\,\mathrm{d}x \sim O(\epsilon^\beta \Psi) \tag{6.8b}$$

for every 'good function' $\Psi(x)$. The 'good functions' are defined in Lighthill (1958).

There are certainly other metrics that can equally well gauge the asymptotic error. Among them, one that is normally used is the metric of continuous functions that asserts a uniform error along the length of the body. This metric will be distinguished by the symbol $\doteq$, as in (6.5), (6.7). It seems, however, that the measure (6.8) is more suitable for analysing the error factor of the present slender-body theory. Some of the reasons are indicated in the Appendix, but the adequacy of this choice is best realized by contrasting theory with numerical results. This will be done at the end of the present paper.

With (6.8) we can easily show (see Appendix) that

(*a*) if $P(x) \in B_{\mathrm{E}}[0; L]$, then

$$\left.\begin{aligned} P(x) &= \overline{P}(x)\left[1+O\left(\frac{1}{\lambda^2}\right)\right], \\ \frac{1}{\lambda^n}\frac{\mathrm{d}^n\overline{P}}{\mathrm{d}x^n} &\sim O\left(\frac{P}{\lambda^2}\right) \quad (n = 2, 3, \ldots). \end{aligned}\right\} \tag{6.9}$$

With (6.9), the outer potential (5.1) can be written as†

$$\Phi_0(x, y) \doteq \left\{\int_{-\infty}^{\infty} \mathrm{d}\xi\left[\overline{Q}(\xi)+\frac{1}{K_0}\overline{M}(\xi)\frac{\partial}{\partial y}\right]G(x-\xi; y)\right\}\left(1+O\left(\frac{1}{\lambda^2}\right)\right); \tag{6.10}$$

(*b*) if $P(x) \in B[0; L]$ then

$$\left.\begin{aligned} P(x) &= \overline{P}(x)\left[1+O\left(\frac{1}{\lambda}\right)\right], \\ \frac{1}{\lambda^n}\frac{\mathrm{d}^n\overline{P}}{\mathrm{d}x^n} &\sim O\left(\frac{P}{\lambda}\right) \quad (n = 1, 2, \ldots), \end{aligned}\right\} \tag{6.11}$$

and

$$\Phi_0(x, y) \doteq \left\{\int_{-\infty}^{\infty} \mathrm{d}\xi\left[\overline{Q}(\xi)+\frac{1}{K_0}\overline{M}(\xi)\frac{\partial}{\partial y}\right]G(x-\xi; y)\right\}\left(1+O\left(\frac{1}{\lambda}\right)\right). \tag{6.12}$$

These relations will be used in the next section.

6.2. *Fourier transforms of some special functions*

We shall list here the Fourier transforms of some special functions. They will be used in the next section.

† Actually the error in (6.10) is of order $1/\lambda^2 \ln \lambda \leqslant O(1/\lambda^\alpha)$ for any $\alpha < 2$. We ignored the logarithm term in the following.

We start with the Green function (2.6). Following Erdélyi (1954) we obtain

$$\left.\begin{aligned} G(x,y) &= -\tfrac{1}{4}\mathrm{i}H_0^{(1)}(K_0(x^2+y^2)^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}xK_0\cos\alpha},\\ G^*(y;K) &= F[G(x,y)] = -\tfrac{1}{2}\mathrm{i}\frac{\exp\left(\mathrm{i}(K_0^2-(K-K_0\cos\alpha)^2)^{\frac{1}{2}}|y|\right)}{(K_0^2-(K-K_0\cos\alpha)^2)^{\frac{1}{2}}}.\end{aligned}\right\} \tag{6.13a}$$

A particular case is given by

$$\left.\begin{aligned} h(x) &= \tfrac{1}{2}H_0^{(1)}(K_0|x|),\\ h^*(K) &= \frac{1}{(K_0^2-K^2)^{\frac{1}{2}}}.\end{aligned}\right\} \tag{6.13b}$$

The third Fourier-transform pair is defined below:

$$\left.\begin{aligned} f(x) &= \begin{cases} 0 & (x<0),\\ \dfrac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\pi^{\frac{1}{2}}}\dfrac{1}{x^{\frac{1}{2}}} & (x>0).\end{cases}\\ f^*(K) &= \frac{1}{K^{\frac{1}{2}}}.\end{aligned}\right\} \tag{6.13c}$$

(In the Fourier transforms (6.13) we assumed $\sqrt{-1} = +\mathrm{i}$.) An important function is also introduced here. It is defined by

$$\left.\begin{aligned} I_P(x;\alpha) &= \mathscr{L}(P) = F^{-1}[P^*(K)\,h^*(K-K_0\cos\alpha)],\\ \mathscr{L}(P) &= \frac{1}{2}\int_0^L \mathrm{d}\xi P(\xi)\,\mathrm{e}^{-\mathrm{i}K_0(x-\xi)\cos\alpha}\,H_0^{(1)}(K_0|x-\xi|).\end{aligned}\right\} \tag{6.14}$$

With the help of (6.9) and (6.11) we can write

$$I_P(x;\alpha) = \{F^{-1}[P^*(K)\,h^*(K-K_0\cos\alpha)]_{|K|\leqslant\lambda}\}(1+O(1/\lambda^2)),\quad P(x)\in B_{\mathrm{E}}[0;L], \tag{6.15a}$$

or

$$I_P(x;\alpha) = \{F^{-1}[P^*(K)\,h^*(K-K_0\cos\alpha)]_{|K|\leqslant\lambda}\}(1+O(1/\lambda)),\quad P(x)\in B[0;L]. \tag{6.15b}$$

We notice that in (6.11), (6.12) and (6.15*b*) the error factor is $O(1/\lambda)$, instead of $O(1/\lambda^2)$ as for a body with pointed ends. As we will see later on, these results suffice for the present theory.

7. The inner expansion of the outer solution

In order to derive the inner expansion of the outer solution we will analyse first the three complementary ranges associated with the classical theories. In this way we will study long waves (Froude–Krilov) in §7.1, short waves with $\sin\alpha \sim O(1)$ (strip theory) in §7.2 and short waves with $\alpha = 0$ (parabolic approximation) in §7.3. We will obtain three distinct asymptotic expressions, and in §7.4 we will derive a uniform expansion, valid for all wavelengths and angles of incidence.

All these results will first be derived for a body with pointed ends – or, more mathematically – under the assumption that $P(x) = \{Q(x);M(x)\}\in B_{\mathrm{E}}[0;L]$. In §7.5 we analyse closely the consistency of the inner expansion so obtained and with this we can easily extend the result so derived to the whole class $B[0;L]$. This extension will be done in §7.6.

7.1. *Long waves* ($K_0 \sim O(1)$)

For long waves the source strength σ, introduced in (2.8), is of order 1. From (6.1 *b*) it follows then that

$$\left.\begin{aligned} Q &\sim O(\epsilon), \\ M &\sim O(\epsilon^2), \end{aligned}\right\} \quad K_0 B \sim O(\epsilon), \tag{7.1}$$

and from (5.1)

$$\Phi_0 \sim O(\epsilon). \tag{7.2}$$

So the scattered potential is of order ϵ and this is just the Froude–Krilov approximation. To keep the overall error at order ϵ^2 it is necessary here to approximate Φ_0 only with an error factor of the form $[1+O(\epsilon)]$. Taking the Fourier transform of (5.1) and using (6.9), (6.10), we obtain

$$\Phi_0^*(y;K) = \left\{\left[Q^*(K)+\frac{1}{K_0}M^*(K)\frac{\partial}{\partial y}\right]G^*(y;K)\right\}(1+O(\epsilon)), \quad |K| \leqslant \lambda \sim O\left(\frac{1}{\epsilon^{\frac{1}{2}}}\right), \tag{7.3}$$

or, using (6.13 *a, b*),

$$\begin{aligned}\Phi_0^*(y;K) = &-\tfrac{1}{2}\mathrm{i}Q^*(K)\,h^*(K-K_0\cos\alpha)\exp\left(\mathrm{i}(K_0^2-(K-K_0\cos\alpha)^2)^{\frac{1}{2}}|y|\right) \\ &+(\operatorname{sign} y)\frac{M^*(K)}{2K_0}\exp\left(\mathrm{i}(K_0^2-(K-K_0\cos\alpha)^2)^{\frac{1}{2}}|y|\right).\end{aligned} \tag{7.4}$$

Since $|K| \leqslant \lambda \sim O(1/\epsilon^{\frac{1}{2}})$ then, in the inner region $|y| \sim O(\epsilon)$, we have

$$\mathrm{i}(K_0^2-(K-K_0\cos\alpha)^2)^{\frac{1}{2}}|y| \leqslant O(\epsilon^{\frac{1}{2}}). \tag{7.5}$$

Developing the exponential of this factor in Taylor's series we obtain, with the help of (6.13 *b*),

$$\exp\left(\mathrm{i}(K_0^2-(K-K_0\cos\alpha)^2)^{\frac{1}{2}}|y|\right) = 1+\mathrm{i}\frac{y}{h^*(K-K_0\cos\alpha)}+O(\epsilon), \tag{7.6}$$

when $|y| \sim O(\epsilon)$.

If we recall that $M \sim O(\epsilon^2)$ in the present case then, inserting (7.6) into (7.4), we obtain

$$\Phi_0^*(y;K) \sim \left\{[-\tfrac{1}{2}\mathrm{i}Q^*(K)\,h^*(K-K_0\cos\alpha)+\tfrac{1}{2}Q^*(K)\,|y|]+(\operatorname{sign} y)\frac{M^*(K)}{K_0}\right\}(1+O(\epsilon)), \tag{7.7}$$

or, taking the inverse Fourier transform,

$$\Phi_0(x,y) \sim \left[-\tfrac{1}{2}\mathrm{i}I(x;\alpha)+\tfrac{1}{2}Q(x;\alpha)\,|y|+(\operatorname{sign} y)\frac{M(x;\alpha)}{2K_0}\right](1+O(\epsilon)) \quad (|y| \sim O(\epsilon)), \tag{7.8}$$

where we have used $I(x;\alpha) = \mathscr{L}(Q)$; see (6.14).†

Expression (7.8) is the inner expansion of the outer solution in the long-wave regime. It has been deduced under the condition that $M \sim O(\epsilon^2)$ which is valid only in shallow water or for a body with shallow draft. In §7.5 we will see why this condition is not really needed.

† To keep the notation short we will write $I(x;\alpha)$ in place of $I_Q(x;\alpha) = \mathscr{L}(Q)$, see (6.14).

7.2. *Short waves*; $\sin\alpha \sim O(1)\,(K_0 \sim O(1/\epsilon))$

If we take $\lambda \sim O(1/\epsilon)$ we obtain, with an error factor of the form $[1+O(\epsilon^2)]$,

$$\left.\begin{aligned}\Phi_0^*(y;K) &= \left[Q^*(K)+\frac{1}{K_0}M^*(K)\frac{\partial}{\partial y}\right]G^*(y;K)\,|_{|K|\leqslant\lambda},\\ I(x;\alpha) &= F^{-1}[Q^*(K)\,h^*(K-K_0\cos\alpha)]_{|K|\leqslant\lambda},\end{aligned}\right\} \tag{7.9}$$

where we have used (6.9), (6.10) and (6.15*a*).

To properly analyse (7.9) in the inner region $|y| \sim O(\epsilon)$ we must deal with the radical

$$\left.\begin{aligned}|K_0^2-(K-K_0\cos\alpha)^2|^{\frac{1}{2}} &= (K_0\sin\alpha)\left|1+f\left(\frac{K}{K_0}\right)\right|^{\frac{1}{2}},\\ f\left(\frac{K}{K_0}\right) &= 2\frac{K}{K_0}\frac{\cos\alpha}{\sin^2\alpha}-\left(\frac{K}{K_0}\right)^2\frac{1}{\sin^2\alpha}\end{aligned}\right\} \tag{7.10}$$

that appears in $G^*(y;K)$ and $h^*(K-K_0\cos\alpha)$; see (6.13).

We notice, however, that if $|f(K/K_0)| < 1$ then we can use Taylor's series expansion for $(1+t)^{\pm\frac{1}{2}}$, $|t| < 1$, to simplify (7.10). The condition $f(K/K_0)\,| < 1$ is fulfilled if

$$|K| < \lambda = K_0(1-\cos\alpha) \sim O(1/\epsilon). \tag{7.11}$$

Using this λ together with Taylor's series expansion of (7.10) we obtain

$$\left.\begin{aligned}h^*(K-K_0\cos\alpha)\,|_{|K|\leqslant\lambda} &= \frac{1}{K_0\sin\alpha}\left[1+\sum_{n=1}^{\infty}\beta_n\left(\frac{K}{K_0}\right)^n\right],\\ \beta_1 &= -\frac{\cos\alpha}{\sin^2\alpha},\quad \beta_n \sim O(1).\end{aligned}\right\} \tag{7.12}$$

We can also write

$$\left.\begin{aligned}G^*(y;K)\,|_{|K|\leqslant\lambda} &= -\tfrac{1}{2}\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}}{K_0\sin\alpha}\left[1+\sum_{n=1}^{\infty}\gamma_n\left(\frac{K}{K_0}\right)^n\right],\\ \gamma_1 &= -\frac{\cos\alpha}{\sin^2\alpha}+\mathrm{i}\frac{\cos\alpha}{\sin\alpha}K_0|y|,\quad \gamma_n \sim O((K_0|y|)^n),\end{aligned}\right\} \tag{7.13}$$

and

$$\left.\begin{aligned}\frac{\partial G^*}{\partial y}(y;K)|_{|K|\leqslant\lambda} &= -\tfrac{1}{2}\mathrm{i}\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\left[1+\sum_{n=1}^{\infty}\rho_n\left(\frac{K}{K_0}\right)^n\right],\\ \rho_1 &= \mathrm{i}\frac{\cos\alpha}{\sin\alpha}K_0|y|,\quad \rho_n \sim O((K_0|y|)^n).\end{aligned}\right\} \tag{7.14}$$

Inserting these relations into (7.9) we get, with an error factor of the form $[1+O(\epsilon^2)]$, the expressions

$$\begin{aligned}\Phi_0(x,y) = {}&-\tfrac{1}{2}\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}}{K_0\sin\alpha}F^{-1}\left[Q^*(K)+\frac{\gamma_1}{K_0}KQ^*(K)+\sum_{n=2}^{\infty}\frac{\gamma_n}{K_0^n}K^nQ^*(K)\right]_{|K|\leqslant\lambda}\\ &+(\operatorname{sign}y)\frac{\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}}{2K_0}F^{-1}\left[M^*(K)+\frac{\rho_1}{K_0}KM^*(K)+\sum_{n=2}^{\infty}\frac{\rho_n}{K_0^n}K^nM^*(K)\right]_{|K|\leqslant\lambda},\end{aligned} \tag{7.15}$$

and

$$I(x;\alpha) = \frac{1}{K_0\sin\alpha}F^{-1}\left[Q^*(K)+\frac{\beta_1}{K_0}KQ^*(K)+\sum_{n=2}^{\infty}\frac{\beta_n}{K_0^n}K^nQ^*(K)\right]_{|K|\leqslant\lambda}. \tag{7.16}$$

We notice, however, that if $P(x) = \{Q(x); M(x)\} \in B_{\mathrm{E}}[0, L]$ then, from (6.9),

$$\frac{1}{K_0^n}\frac{\mathrm{d}^n\overline{P}}{\mathrm{d}x^n} = F^{-1}\left[(-\mathrm{i})^n\left(\frac{K}{K_0}\right)^n P^*(K)\right]_{|K|\leqslant\lambda} \sim O(\epsilon^2 P) \quad (n \geqslant 2), \tag{7.17}$$

since $K_0 \sim O(\lambda) \sim O(1/\epsilon)$.

In the *inner region* ($|y| \sim O(\epsilon)$), $K_0|y| \sim O(1)$ and so $\{\beta_n, \lambda_n, \rho_n\} \sim O(1)$, all n; see (7.12), (7.13), (7.14). From this fact and (7.17) we obtain

$$\begin{aligned}\Phi_0(x,y) = \Big\{-\tfrac{1}{2}\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}}{K_0\sin\alpha}\left[Q(x;\alpha) - \left(\mathrm{i}\frac{\cos\alpha}{\sin^2\alpha} + \frac{\cos\alpha}{\sin\alpha}K_0|y|\right)\frac{1}{K_0}\frac{\mathrm{d}Q}{\mathrm{d}x}(x;\alpha)\right] \\ + (\operatorname{sign} y)\frac{\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}}{2K_0}\left[M(x;\alpha) - \frac{\cos\alpha}{\sin\alpha}|y|\frac{\mathrm{d}M}{\mathrm{d}x}(x;\alpha)\right]\Big\}(1+O(\epsilon^2)) \\ \text{when } |y| \sim O(\epsilon),\end{aligned} \tag{7.18}$$

and also

$$I(x;\alpha) = \frac{1}{K_0\sin\alpha}\left[Q(x;\alpha) - \mathrm{i}\frac{\cos\alpha}{\sin^2\alpha}\frac{1}{K_0}\frac{\mathrm{d}Q}{\mathrm{d}x}(x;\alpha)\right](1+O(\epsilon^2)). \tag{7.19}$$

In the above expressions we have used the relation

$$\{\overline{Q}(x); \overline{M}(x)\} = \{Q(x); M(x)\}(1+O(\epsilon^2));$$

see (6–9).

From (7.18) and (7.19) we obtain also

$$\left.\begin{aligned}\Phi_0(x,y) &= \mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\left[-\frac{\mathrm{i}Q(x;\alpha)}{2K_0\sin\alpha} + (\sin y)\frac{M(x;\alpha)}{2K_0}\right] \quad (1+O(\epsilon)), \\ I(x;\alpha) &= \frac{Q(x;\alpha)}{K_0\sin\alpha}(1+O(\epsilon)).\end{aligned}\right\} \tag{7.20}$$

But (7.20) is just the radiation condition (2.11) for the cross-section problem, with $T = \frac{1}{2}(-\mathrm{i}(Q(x;\alpha)/K_0\sin\alpha) + (M/K_0))$ and $R = \frac{1}{2}(-\mathrm{i}(Q(x;\alpha)/K_0\sin\alpha) - (M/K_0))$. So strip theory is correct with an error factor $(1+O(\epsilon))$.

Expression (7.18) is the inner expansion of the outer solution when the waves are short and $\sin\alpha \sim O(1)$. In order to obtain an expression more similar to (7.8) we can compute $Q(x;\alpha)/K_0\sin\alpha$ from (7.19) and use this value in (7.18). If we define then

$$\begin{aligned}\Phi_0^{(1)}(x,y) = \left[-\tfrac{1}{2}\mathrm{i}I(x;\alpha)\cos(K_0|y|\sin\alpha) + \frac{1}{2}\frac{Q(x;\alpha)}{K_0\sin\alpha}\sin(K_0|y|\sin\alpha)\right] \\ + (\operatorname{sign} y)\frac{M(x;\alpha)}{2K_0}\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha},\end{aligned} \tag{7.21a}$$

$$\begin{aligned}\Phi_0^{(2)}(x,y) = -\tfrac{1}{2}\mathrm{i}\left[\frac{\cos\alpha}{K_0\sin^2\alpha}\sin(K_0|y|\sin\alpha) - \frac{\cos\alpha}{\sin\alpha}|y|\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\right]\frac{\mathrm{d}Q/\mathrm{d}x}{K_0\sin\alpha} \\ + (\operatorname{sign} y)\left[-\frac{\cos\alpha}{\sin\alpha}|y|\frac{1}{2K_0}\frac{\mathrm{d}M}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\right],\end{aligned} \tag{7.21b}$$

we obtain

$$\left.\begin{aligned}&\Phi_0(x,y) \sim [\Phi_0^{(1)}(x,y) + \Phi_0^{(2)}(x,y)](1+O(\epsilon^2)) \quad (|y| \sim O(\epsilon)), \\ &\frac{\Phi_0^{(2)}}{\Phi_0^{(1)}} \sim O(\epsilon).\end{aligned}\right\} \tag{7.22}$$

This inner expansion of the outer solution is equivalent to (7.18), although more convenient. In fact, since $\sin\alpha \sim O(1)$, we can recover (7.8) if we let $K_0 \sim O(1)$ in (7.21) and (7.22).

7.3. *Short waves*; $\alpha = 0$ ($K_0 \sim O(1/\epsilon)$)

In this case we obtain, with an error $[1+O(\epsilon^2)]$,

$$\left.\begin{aligned} I(x;0) &= F^{-1}\left[\frac{Q^*(K)}{(2KK_0-K^2)^{\frac{1}{2}}}\right]_{|K|\leqslant\lambda}, \\ \Phi_0(x,y;0) &= F^{-1}\left[-\tfrac{1}{2}\mathrm{i}Q^*(K)\frac{\exp(\mathrm{i}(2KK_0-K^2)^{\frac{1}{2}}|y|)}{(2KK_0-K^2)^{\frac{1}{2}}}\right]_{|K|\leqslant\lambda} \end{aligned}\right\} \tag{7.23}$$

for $\lambda \sim O(1/\epsilon)$. In (7.23) we implicitly assumed $M(x;0) = 0$ since we will prove later that†

$$M(x;\alpha) \sim O(\sin\alpha) \quad (\alpha \to 0). \tag{7.24}$$

Taking $\lambda = 2K_0$ we can easily check that

$$I(x;0) = F^{-1}\left[\frac{Q^*(K)}{(2KK_0)^{\frac{1}{2}}}\left(1+\frac{1}{2}\frac{K}{K_0}+\dots\right)\right]_{|K|\leqslant\lambda}. \tag{7.25}$$

If we now define the function (see (6.13c))

$$J(x) = F^{-1}\left[\frac{Q^*(K)}{(2KK_0)^{\frac{1}{2}}}\right] = \frac{1-\mathrm{i}}{2(\pi K_0)^{\frac{1}{2}}}\int_0^x \mathrm{d}\xi\,\frac{Q(\xi;0)}{(x-\xi)^{\frac{1}{2}}}, \tag{7.26}$$

we obtain, from (7.25),

$$I(x;0) = J(x)\,(1+O(\epsilon)) \sim O(\epsilon^{\frac{1}{2}}Q). \tag{7.27}$$

These relations will be used a little later. Expanding $\exp(\mathrm{i}(2KK_0-K^2)^{\frac{1}{2}}|y|)$ in power series in (7.23) we get

$$\begin{aligned} \Phi_0(x,y) = -\tfrac{1}{2}\mathrm{i}F^{-1}\bigg[&\frac{Q^*(K)}{(2KK_0-K^2)^{\frac{1}{2}}}+\mathrm{i}|y|Q^*(K)-\tfrac{1}{2}|y|^2 \\ &\times\frac{Q(K)}{(2KK_0-K^2)^{\frac{1}{2}}}(2KK_0-K_0^2)-\tfrac{1}{6}\mathrm{i}|y|^3Q^*(K)\,(2KK_0-K^2)-\dots\bigg]_{|K|\leqslant\lambda} \end{aligned}$$

or

$$\begin{aligned} \Phi_0(x,y) = -\tfrac{1}{2}\mathrm{i}I(x;0)+\tfrac{1}{2}|y|Q(x;0)-\tfrac{1}{2}K_0|y|^2\frac{\mathrm{d}I}{\mathrm{d}x}(x;0)-\tfrac{1}{6}\mathrm{i}K_0|y|^3\frac{\mathrm{d}Q}{\mathrm{d}x}(x;0) \\ -\tfrac{1}{4}\mathrm{i}|y|^2\frac{\mathrm{d}^2I}{\mathrm{d}x^2}(x;0)-\tfrac{1}{12}|y|^3\frac{\mathrm{d}^2Q}{\mathrm{d}x^2}(x;0)+\dots. \end{aligned} \tag{7.28}$$

The leading term is $O(\epsilon^{\frac{1}{2}}Q)$, see (7.27). It follows, then, that the two last terms in (7.28) are, in the inner region, of relative order ϵ^2 and $\epsilon^{\frac{5}{2}}$, respectively, and can be disregarded.

As a conclusion the inner expansion of the outer solution can again be written in the form (7.22) where, now,

$$\Phi_0^{(1)}(x,y) = -\tfrac{1}{2}\mathrm{i}I(x;0)+\tfrac{1}{2}|y|\,Q(x;0), \tag{7.29a}$$

$$\Phi_0^{(2)}(x,y) = -\tfrac{1}{2}K_0|y|^2\frac{\mathrm{d}I}{\mathrm{d}x}(x;0)-\tfrac{1}{6}\mathrm{i}K_0|y|^3\frac{\mathrm{d}Q}{\mathrm{d}x}(x;0). \tag{7.29b}$$

† Relation (7.24) is valid also for a non-symmetric cross-section. This is another peculiarity of the head-sea diffraction.

We should notice that (7.29*a*) agrees with (7.8) when $\alpha = 0$, since $M(x;\alpha) \to 0$ when $\alpha \to 0$; see (7.24). Also, from (7.27) and (7.29*a*), we obtain

$$\Phi_0(x,y) \sim [-\tfrac{1}{2}\mathrm{i}J(x) + \tfrac{1}{2}|y|\,Q(x;0)]\,(1+O(\epsilon)),$$

and this is just the inner expansion of the parabolic approximation; see (3.3*a*).

7.4. *The uniform expansion*

Since (7.8) can be recovered from (7.21) ($\sin\alpha \sim O(1)$) or from (7.29) ($\alpha = 0$), when $K_0 \sim O(1)$, it remains to check the compatibility between (7.21) and (7.29).

To leading order we can easily see that

$$\lim_{\alpha\to 0} \Phi_0^{(1)}(x,y;\alpha) = \Phi_0^{(1)}(x,y;0),$$

where the first expression is given by (7.21*a*) and the second by (7.29*a*). If we take, however, the limit of (7.21*b*) when $\alpha \to 0$, we obtain

$$\Phi_0^{(2)}(x,y;\alpha) \sim \left\{\left[-\tfrac{1}{2}K_0|y|^2\frac{\mathrm{d}Q/\mathrm{d}x}{K_0\sin\alpha} - \tfrac{1}{6}\mathrm{i}K_0|y|^3\frac{\mathrm{d}Q}{\mathrm{d}x}\right] + (\sin y)\left[-\tfrac{1}{2}|y|\frac{\mathrm{d}M/\mathrm{d}x}{K_0\sin\alpha}\right]\right\} \quad (\alpha\to 0). \tag{7.30}$$

From (7.24), $(K_0\sin\alpha)^{-1}\,\mathrm{d}M/\mathrm{d}x$ is bounded when $\alpha \to 0$ and, as we are going to see later on, this term is actually zero for an 'almost uniform' cross-section. It remains to analyse the even part of (7.30), but we observe that the place of the singular term $(K_0\sin\alpha)^{-1}\,\mathrm{d}Q/\mathrm{d}x$ is taken by $\mathrm{d}I/\mathrm{d}x$ in (7.29*b*).

Expression (7.21) has been deduced, however, for $\sin\alpha \sim O(1)$ and in this case (7.20) suggests the relation

$$\frac{\mathrm{d}I}{\mathrm{d}x}(x;\alpha) = \frac{\mathrm{d}Q/\mathrm{d}x}{K_0\sin\alpha}(1+O(\epsilon)) \quad \sin\alpha \sim O(1). \tag{7.31}$$

In reality we can demonstrate (7.31) *for all* $Q(x) \in B[0;L]$. This demonstration is given in the Appendix and it uses explicitly the metric introduced in (6.8). From (7.31) it follows that the function

$$\Delta\Phi_0^{(2)}(x,y;\alpha) = -\tfrac{1}{2}\mathrm{i}y\frac{\cos\alpha}{\sin\alpha}\sin(K_0|y|\sin\alpha)\left(\frac{\mathrm{d}Q/\mathrm{d}x}{K_0\sin\alpha} - \frac{\mathrm{d}I}{\mathrm{d}x}\right),$$

is of order ϵ^2 when $\sin\alpha \sim O(1)$. Adding this expression to (7.21*b*) we obtain a new $\Phi_0^{(2)}(x,y;\alpha)$ not only continuous in α but that also agrees with (7.29) in the limit $\alpha \to 0$.

In order to write the uniform expansion we recall that, in fact, $Q(\xi) = Q_1(\xi) + Q_2(\xi)$ etc.; see (5.1). It follows then that

$$\left.\begin{aligned} &\Phi_0(x,y;\alpha) \sim [\Phi_0^{(1)}(x,y;\alpha) + \Phi_0^{(2)}(x,y;\alpha)]\,(1+O(\epsilon^2)) \quad (|y| \sim O(\epsilon)),\\ &\frac{\Phi_0^{(2)}}{\Phi_0^{(1)}} \sim O(\epsilon),\\ &\Phi_0^{(2)}(x,y;\alpha) = \Phi_{0,\mathrm{P}}^{(2)}(x,y;\alpha) + \Phi_{0,\mathrm{H}}^{(2)}(x,y;\alpha),\end{aligned}\right\} \tag{7.32a}$$

where

$$\Phi_0^{(1)}(x,y;\alpha) = \left[-\tfrac{1}{2}\mathrm{i}I_1(x;\alpha)\cos(K_0|y|\sin\alpha) + \frac{1}{2}\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\sin(K_0|y|\sin\alpha)\right] + (\operatorname{sign} y)\left[\frac{M_1(x;\alpha)}{2K_0}\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\right], \tag{7.32b}$$

$$\Phi_{0,\mathrm{H}}^{(2)}(x,y;\alpha) = \left[-\tfrac{1}{2}\mathrm{i}I_2(x;\alpha)\cos(K_0|y|\sin\alpha) + \frac{1}{2}\frac{Q_2(x;\alpha)}{K_0\sin\alpha}\sin(K_0|y|\sin\alpha)\right] + (\operatorname{sign} y)\left[\frac{M_2(x;\alpha)}{2K_0}\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\right], \tag{7.32c}$$

$$\begin{aligned}\Phi_{0,\mathrm{P}}^{(2)}(x,y;\alpha) = -\tfrac{1}{2}\mathrm{i}\Bigg[\left(\left(\frac{\cos\alpha}{K_0\sin^2\alpha}\right)\sin(K_0|y|\sin\alpha) - \frac{\cos\alpha}{\sin\alpha}|y|\cos(K_0|y|\sin\alpha)\right)\frac{\mathrm{d}Q/\mathrm{d}x_1}{K_0\sin\alpha} \\ -\mathrm{i}\frac{\cos\alpha}{\sin\alpha}|y|\sin(K_0|y|\sin\alpha)\frac{\mathrm{d}I_1}{\mathrm{d}x}(x;\alpha)\Bigg] \\ + (\operatorname{sign} y)\left[-|y|\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\frac{\cos\alpha}{\sin\alpha}\frac{\mathrm{d}M_1/\mathrm{d}x}{2K_0}\right].\end{aligned} \tag{7.32d}$$

In the above expression we have used the notation $I_j(x;\alpha) = \mathscr{L}(Q_j)$, $j = 1,2$; see (6.14). Obviously $\Phi_{0,\mathrm{H}}^{(2)}$ coincides with $\Phi_0^{(1)}$ with Q_2 in place of Q_1.

7.5. *Consistency of the inner expansion*

Expression (7.32) shows the nature of the outer potential in the overlap region $|y| \sim O(\epsilon)$. In this part of the fluid domain, however, the field equation can be approximated by (4.3), (4.4) with $|y| \geqslant b(x)$. We must then have

$$\frac{\partial^2\Phi_0^{(1)}}{\partial y^2} + (K_0\sin\alpha)^2\Phi_0^{(1)} = 0, \tag{7.33a}$$

$$\frac{\partial^2\Phi_0^{(2)}}{\partial y^2} + (K_0\sin\alpha)^2\Phi_0^{(2)} = -2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi_0^{(1)}}{\partial x}. \tag{7.33b}$$

It is an easy task to confirm the validity of (7.33) and this shows quite clearly the consistency of the inner expansion (7.32).

We observe also that the most general solution of (7.33b) is the sum of a homogeneous with a particular solution. The term $\Phi_{0,\mathrm{H}}^{(2)}(x,y;\alpha)$, see (7.32$c$), is just the homogeneous solution and $\Phi_{0,\mathrm{P}}^{(2)}(x,y;\alpha)$ is the particular one, see (7.32d). Furthermore $\Phi_{0,\mathrm{P}}^{(2)}(x,y;\alpha)$ is uniquely determined by the leading term $\Phi_0^{(1)}(x,y;\alpha)$ since its only homogeneous contribution

$$\tfrac{1}{2}\mathrm{i}\frac{\cos\alpha}{K_0\sin^2\alpha}\sin(K_0\sin\alpha|y|)\frac{\mathrm{d}Q_1/\mathrm{d}x}{K_0\sin\alpha},$$

is just needed for the particular solution in head-sea.

We must notice, also, that (4.3) and (4.4) have been derived from the basic assumption of the slender-body theory; namely, if $f = f(x,y)$ then†

$$\frac{\partial f}{\partial x} \sim O(f) \quad (O^+ \leqslant x \leqslant L^-), \tag{7.34a}$$

$$\frac{\partial f}{\partial y} \sim O\left(\frac{f}{\epsilon}\right). \tag{7.34b}$$

Expression (7.32), on the other hand, has been derived from (7.34*a*) and the explicit use of the metric (6.8). In the Appendix we show that, in fact, (7.34*a*) is in general true only in the metric (6.8). This circumstance certainly reinforces our belief that the asymptotic error must be measured in the metric of the distribution theory.

From (7.33*b*) and (7.34) we can formally derive (4.8*a*) and from this last expression we obtain that $\Phi^{(2)}/\Phi^{(1)} \sim O(\epsilon^2)$ when $K_0 \sim O(1)$. In this case, however, $\Phi^{(1)} \sim O(\epsilon)$ and so $\Phi^{(2)} \sim O(\epsilon^3)$. Or, in other words, *for long waves the leading-order term is in fact correct with an error factor of the form* $[1+O(\epsilon^3)]$. This result is certainly independent of the condition $M \sim O(\epsilon^2)$, as has been pointed out at the end of §7.1.

7.6 *Body with blunt ends*

In this case, using (6.11), (6.12) and (6.15*b*), we obtain, to leading order, the asymptotic approximation

$$\Phi_0(x,y;\alpha) \sim \Phi_0^{(1)}(x,y;\alpha)(1+O(\epsilon)) \quad (|y| \sim O(\epsilon)), \tag{7.35}$$

where $\Phi_0^{(1)}(x,y;\alpha)$ is again given by (7.32*b*).

To obtain the second-order term $\Phi_0^{(2)}(x;y;\alpha)$ we should use a more precise expression for $P(x)$ than the one indicated in (6.5), but the analysis here is much more complicated. It seems more expedient to observe that the inner equations (7.33) continue to be valid provided that the ends are not too close, namely, if $\{x; L-x\} \geqslant O(\epsilon)$. In this region $\Phi_0^{(2)}(x;y;\alpha)$ is a solution of (7.33*b*) and so it must be given by (7.32*a, c, d*) as we have seen in §7.5.

It remains to determine the error in the neighbourhood $O(\epsilon)$ of the ends, but the metric (6.8) seems to indicate that this local error should be of order ϵ. This result can be formalized. In fact, if $\Phi_e(x,y;\alpha)$ is the exact solution and $\Phi_a(x,y;\alpha) = \Phi^{(1)}(x,y;\alpha)+\Phi^{(2)}(x,y;\alpha)$ its asymptotic approximation, then consider the error $\Delta\Phi(x,y;\alpha) = \Phi_e(x,y;\alpha)-\Phi_a(x,y;\alpha)$. But, from (7.34*a*), $\{(\partial\Phi_e/\partial x); (\partial\Phi_a/\partial x)\} \sim O(1)$ and so $\Delta\Phi(x,y;\alpha)$ can increase only by a factor of order ϵ in the neighbourhood ϵ of the ends. So‡

$$\Delta\Phi(x,y;\alpha) = \Phi_e(x,y;\alpha)-\Phi_a(x,y;\alpha) \sim \begin{cases} \sim O(\epsilon) & (\{x;L-x\} \leqslant O(\epsilon)) \\ \sim O(\epsilon^2) & (\{x;L-x\} \sim O(1)). \end{cases} \tag{7.36}$$

That is, $\Delta\Phi(x,y;\alpha)$ is of order ϵ^2 in the metric (6.8). The error behaviour (7.36) is confirmed by the numerical experiments.

The inner expansion of the outer solution is then given by (7.32) for all bodies with form function $g(x) \in B[0;L]$; see (6.2).

† We have avoided the points $x = 0$; $x = L$ since $\mathrm{d}f/\mathrm{d}x$ can contain a Dirac function there. See, for instance, the form function $g(x)$ for a body with blunt ends.

‡ *Note added in proof*: relation (7.36) is not strictly valid for very short waves ($K_0 \gg 1/\epsilon$). See §6 of Paper II for details.

8. Matching

To determine the functions $\{S_1(x);\ A_1(x);\ S_2(x);\ A_2(x)\}$ of the inner solution (see (4.5), (4.7)) and the densities $\{Q_1(x);\ M_1(x);\ Q_2(x);\ M_2(x)\}$ of the outer solution (see (5.1)) we must match the inner expansion of the outer solution (see (7.32)) with the outer expansion of the inner solution, given by (2.16), (4.5) and (4.7). We will do this in two steps.

8.1. *Leading-order matching*

Matching (4.5) (see also (2.16)) with (7.32*b*) we obtain

$$\left.\begin{aligned} \tfrac{1}{2}\mathrm{i}I_1(x;\alpha) &= (1-H)[1-S_1(x;\alpha)(R_{\mathrm{S}}(\alpha)+1)]+H\left[1-US_1(x;0)\left(\frac{1}{A_1}-K_0 b\right)\right], \\ (1-H)\,\mathrm{i}S_1(x;\alpha)R_{\mathrm{S}}(\alpha)+HS_1(x;0)\,UK_0 &= (1-H)\frac{1}{2}\frac{Q_1(x;\alpha)}{K_0\sin\alpha}+H\tfrac{1}{2}Q_1(x;0), \end{aligned}\right\} \tag{8.1}$$

and

$$\left.\begin{aligned} A_1(x;\alpha)\,R_{\mathrm{A}}(\alpha) &= \frac{M_1(x;\alpha)}{2K_0}, \\ A_1(x;\alpha)(R_{\mathrm{A}}(\alpha)+1) &= 1+\frac{M_1(x;\alpha)}{2K_0}. \end{aligned}\right\} \tag{8.2}$$

From (8.2) we get†

$$\left.\begin{aligned} A_1(x;\alpha) &= 1, \\ \frac{M_1(x;\alpha)}{2K_0} &= R_{\mathrm{A}}(\alpha), \end{aligned}\right\} \tag{8.3}$$

and from (8.1)

$$\left.\begin{aligned} S_1(x;\alpha) &= -\tfrac{1}{2}\mathrm{i}\,\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\,\frac{1}{R_{\mathrm{S}}(\alpha)} \quad (0<\alpha\leqslant\tfrac{1}{2}\pi), \\ S_1(x;\alpha)\,U &= \frac{Q_1(x;0)}{2K_0} \quad (\alpha=0), \end{aligned}\right\} \tag{8.4}$$

where $Q_1(x;\alpha)$ is the solution of the integral equation

$$\left.\begin{aligned} \tfrac{1}{2}\mathrm{i}I_1(x;\alpha) &= 1+(1-H)\,\tfrac{1}{2}\mathrm{i}\,\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\,\frac{1+R_{\mathrm{S}}(\alpha)}{R_{\mathrm{S}}(\alpha)}-H\frac{1}{2}\frac{Q_1(x;0)}{K_0}\left(\frac{1}{A_1}-K_0 b\right), \\ I_1(x;\alpha) &= \mathscr{L}(Q_1). \end{aligned}\right\} \tag{8.5}$$

Both (8.4) and (8.5) seem singular in the limit $\alpha\to 0$. If we use, however, (2.19) in (8.4), (8.5), and (2.20) in (4.5) we can easily check the continuity of the slender-body theory in the incidence angle. In particular (2.19) and (8.3) confirm that $M(x;\alpha)\to 0$ as $\alpha\to 0$; see (7.24).

To leading order we have then

$$\left.\begin{aligned} \Phi_{\mathrm{T}}^{(1)}(x,y;\alpha) &= \phi_{\mathrm{T}}(y;\alpha)+[S_1(x;\alpha)-1]\,\phi_{\mathrm{T,S}}(y;\alpha), \\ \phi_{\mathrm{T}}(y;\alpha) &= \text{cross-section total potential} = \phi_{\mathrm{T,S}}(y;\alpha)+(\sin y)\,\phi_{\mathrm{T,A}}(y;\alpha), \end{aligned}\right\} \tag{8.6}$$

† In shallow water, $R_{\mathrm{A}}(\alpha)\sim O(\epsilon^2)$ if the waves are long, see (2.18*b*). This confirms that $M\sim O(\epsilon^2)$ in this case; see §7.1.

where

$$\left.\begin{aligned} S_1(x;\alpha) &= -\tfrac{1}{2}\mathrm{i}\frac{Q_1(x;\alpha)}{K_0 \sin\alpha}\frac{1}{R_S(\alpha)}, \\ \tfrac{1}{2}\mathrm{i}I_1(x;\alpha) &= 1+\tfrac{1}{2}\mathrm{i}\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\frac{1+R_S(\alpha)}{R_S(\alpha)}, \end{aligned}\right\} \tag{8.7}$$

with $I_1(x;\alpha) = \mathscr{L}(Q_1)$; see (6.14).

From (8.3), (8.7) we can easily check that $\{Q(x);\, M(x)\}$ have the same singularities as $\{R_S(\alpha);\, R_A(\alpha)\}$ and so of the form function $g(x) = 2b(x)/B$. Furthermore, since $\{R_S(\alpha);\, R_A(\alpha)\} = 0$ if $b = 0$ (see (2.18), (8.1)), then from (8.3), (8.7) we obtain that $\{Q(x);\, M(x)\} = 0$ in this case. Or, in other words, $\{Q(x);\, M(x)\} \in B[0;L]$ (or $B_E[0;L]$) if $g(x) \in B[0;L]$ (or $B_E[0;L]$). This result confirms the assumption made about the class of functions in which both $\{Q(x);\, M(x)\}$ reside.

8.2. *Second-order term for a uniform cross-section*

To determine the second-order solution we must compute the particular solution (4.6) and match (4.7) with (7.32*c, d*). This matching is trivial now, since $\Phi_0^{(2)}(x,y;\alpha)$ is already a solution of the inner equation (7.33*b*). In what follows, however, we shall demonstrate that for a uniform cross-section the potential $\Phi_0^{(2)}(x,y;\alpha)$ is of order ϵ^2, when the waves are short, and can be disregarded.

In fact, for a uniform cross-section

$$\frac{\partial \phi_T}{\partial x}(y;\alpha) = \frac{\mathrm{d}R_S}{\mathrm{d}x} = \frac{\mathrm{d}R_A}{\mathrm{d}x} = 0,$$

and so (see (8.3), (8.6), (8.7))

$$\left.\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}x}(x;\alpha) &= 0, \\ \frac{\partial \Phi^{(1)}}{\partial x} &= \frac{\mathrm{d}S_1}{\mathrm{d}x}(x;\alpha)\,\phi_{T,S}(y;\alpha), \\ \frac{\mathrm{d}S_1}{\mathrm{d}x}(x;\alpha) &= -\tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}Q_1/\mathrm{d}x}{K_0\sin\alpha}\frac{1}{R_S(\alpha)}, \end{aligned}\right\} \tag{8.8}$$

where

$$\tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}I_1}{\mathrm{d}x}(x;\alpha) = \tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}Q_1/\mathrm{d}x}{K_0\sin\alpha}\frac{1+R_S(\alpha)}{R_S(\alpha)}. \tag{8.9}$$

In the Appendix we show that for short waves and an arbitrary $P(x) \in B[0;L]$ we have

$$\left.\begin{aligned} \frac{\mathrm{d}I_P}{\mathrm{d}x}(x;\alpha) &= \frac{\mathrm{d}P/\mathrm{d}x}{K_0\sin\alpha}(1+O(\epsilon)) \quad (0^+ \leqslant x \leqslant L^-,\ \sin\alpha \sim O(1)), \\ \frac{\mathrm{d}I_P}{\mathrm{d}x}(x;\alpha) &= \left[\frac{1-\mathrm{i}}{2(\pi K_0)^{\frac{1}{2}}}\int_0^x \mathrm{d}\xi\,\frac{\mathrm{d}P/\mathrm{d}\xi}{(x-\xi)^{\frac{1}{2}}}\right](1+O(\epsilon)) \quad (\alpha = 0), \end{aligned}\right\} \tag{8.10}$$

where $I_P(x;\alpha) = \mathscr{L}(P)$; see (6.14).

Placing (8.10) into (8.9) we obtain, with the help of (8.8),†

$$\left.\begin{aligned} -\tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}Q/\mathrm{d}x}{K_0\sin\alpha}\frac{1}{R_S(\alpha)} &\sim O(\epsilon) \\ \frac{\partial\Phi^{(1)}}{\partial x} &\sim O(\epsilon)\end{aligned}\right\}; K_0 B \sim O(1). \tag{8.11}$$

From (4.8a) (see also (7.33b) and (7.34)) we have

$$\Phi^{(2)}(x,y;\alpha) \sim O\left(\epsilon K_0 B\frac{\partial\Phi^{(1)}}{\partial x}\right).$$

Using (8.11) in the above relation we obtain finally

$$\Phi^{(2)}(x,y;\alpha) \sim O(\epsilon^2). \tag{8.12}$$

Or, in short, *for a uniform cross-section the leading-order term $\Phi^{(1)}(x,y;\alpha)$ is correct with an error factor of the form $[1+O(\epsilon^2)]$.*

This important conclusion motivated us to look for a broader class of geometries where the same result also holds true. This class is defined below.

8.3. '*Almost uniform*' *cross-section*

We should hope to extend (8.12) for bodies close, in some sense, to one that has a uniform cross-section. Since the geometry of the body is described by the form function $g(x)$ and the exciting term in (7.33b) is proportional to $\mathrm{d}g/\mathrm{d}x$, it is convenient first to define these two functions for a body with uniform cross-section. In this case we have

$$g_u(x) = \begin{cases} 1 & (0^+ \leqslant x \leqslant L^-), \\ 0 & (x \leqslant 0^- \quad \text{or} \quad x \geqslant L^+). \end{cases} \tag{8.13}$$

The derivative of $g_u(x)$ exists only in the sense of generalized functions and it is given by (see (6.8))

$$\mathrm{D}g_u(\Psi) = \int_{-\infty}^{\infty}\frac{\mathrm{d}g_u}{\mathrm{d}x}(x)\,\Psi(x)\,\mathrm{d}x = \Psi(0)-\Psi(L). \tag{8.14}$$

Looking closely at the expression for $\Phi^{(1)}(x,y;\alpha)$ it is not difficult to see that

$$\frac{\partial\Phi^{(1)}}{\partial x}(x,y;\alpha) = F_\alpha(K_0x, K_0y; g(x))\frac{\mathrm{d}g}{\mathrm{d}x}(x), \tag{8.15}$$

where the function $F_\alpha(.\,,.\,;\,.)$ is independent of the geometry.‡ But in general the derivative of $g(x)$ exists only as a generalized function and it is more precise to write

$$\frac{\partial\Phi^{(1)}}{\partial x}(x,y;\alpha) = F_\alpha(K_0x, K_0y; g(x))\,\mathrm{D}g(\Psi). \tag{8.16}$$

If now $g(x)$ is such that

$$\Delta g(\Psi) = \int_{-\infty}^{\infty}[g(x)-g_u(x)]\,\Psi(x)\,\mathrm{d}x \sim O(\epsilon), \tag{8.17a}$$

$$\mathrm{D}g(\Psi) = \Psi(0)-\Psi(L)+O(\epsilon) = \mathrm{D}g_u(\Psi)+O(\epsilon), \tag{8.17b}$$

† In reality $\partial\Phi^{(1)}/\partial x \sim O(1/K_0)$ when $K_0 \gg 1$. So (8.11) holds even for very short waves ($K_0 \gg 1/\epsilon$).

‡ For example: $A_1 = \tanh(K_0 b(x))$, see (2.14), and so $(\mathrm{d}A_1/\mathrm{d}x)(x) = F(\xi; K_0B)|_{\xi=g(x)}\,(\mathrm{d}g/\mathrm{d}x)(x)$ where $F(\xi; K_0B) = \tfrac{1}{2}K_0B\cosh^{-2}(K_0B\xi)$, independent of $g(x)$.

then, in the metric (6.8),

$$\left.\frac{\partial\phi^{(1)}}{\partial x}\right|_{g(x)} = F_\alpha(K_0 x, K_0 y; g(x))\,\mathrm{D}g(\Psi)$$

$$= [F_\alpha(K_0 x, K_0 y; g_u(x)) + O(\epsilon)]\,(\mathrm{D}g_u(\Psi) + O(\epsilon))$$

$$= F_\alpha(K_0 x, K_0 y; g_u(x))\,\mathrm{D}g_u(\Psi) + O(\epsilon),$$

or

$$\left.\frac{\partial\Phi^{(1)}}{\partial x}\right|_{g(x)} = \left.\frac{\partial\Phi^{(1)}}{\partial x}\right|_{g_u(x)} + O(\epsilon) \sim O(\epsilon), \tag{8.18}$$

where we have used (8.11) to write

$$\left.\frac{\partial\Phi^{(1)}}{\partial x}\right|_{g_u(x)} \sim O(\epsilon).$$

A body is said to have an 'almost uniform' cross-section if its form function $g(x)$ satisfies (8.17). In this case $\partial\Phi^{(1)}/\partial x \sim O(\epsilon)$, see (8.18), and then, from (4.8a) $\Phi^{(2)}(x, y; \alpha) \sim O(\epsilon^2)$.

It remains, here, to characterize, in a clearer way, geometries that satisfy (8.17). We introduce, then, the class $B_u[0; L]$ of functions $g(x)$ such that:

(i) $g(x)$ is continuous, integrable and of order 1 in the interval $0^+ \leqslant x \leqslant L^-$,

(ii) $g(x) \equiv 0$ if $x \leqslant 0^-$ or $x \geqslant L^+$,

(iii) $g(x) = 1 + \Delta_1 \bar{g}(x)$ when $\Delta \leqslant x \leqslant L - \Delta$, where both Δ and Δ_1 are of order ϵ and $\bar{g}(x)$ is continuous with derivative of order 1 in $\Delta \leqslant x \leqslant L - \Delta$.

(iv) besides restriction (i), $g(x)$ is arbitrary in the intervals $0^+ \leqslant x \leqslant \Delta$ and $L - \Delta \leqslant x \leqslant L^-$.

It is not difficult to check that if $g(x) \in B_u[0; L]$ then conditions (8.17) are satisfied and so $\Phi^{(2)} \sim O(\epsilon^2)$. It is also quite clear that the slender geometries used in the off-shore industry are in the class $B_u[0; L]$. For such bodies, then, the leading term is correct with an error factor $1 + O(\epsilon^2)$. It is important here to emphasize the role played by the metric (6.8) in the formalization of the present result.

9. Numerical experiments

In order to check the theory we analysed the geometries indicated in figure 2, with $\epsilon = B/L = 0.20$.

To determine the 'exact solution' we solved numerically the integral equation (2.8) of the full linear theory. Using successive discretizations we could extrapolate a numerical error of order 0.3 %. Discrepancies below this value should not be considered.

To present the results in an easier way we divided the body into 16 sections, section 0 being placed at $x = 0$ and section 16 at $x = L = 1$. In each section we computed

$$\left.\begin{aligned} H(x; \alpha) &= \int_{-b(x)}^{b(x)} \Phi_{\mathrm{T}}(x, y; \alpha)\,\mathrm{d}y, \\ M(x; \alpha) &= \int_{-b(x)}^{b(x)} y\Phi_{\mathrm{T}}(x, y; \alpha)\,\mathrm{d}y. \end{aligned}\right\} \tag{9.1}$$

Obviously $H(x; \alpha)$ is the average value of the symmetric part of Φ_{T} and it is associated with the sectional heave force. $M(x; \alpha)$ is related to the roll moment and it is the weighted average of the antisymmetric part of Φ_{T}.

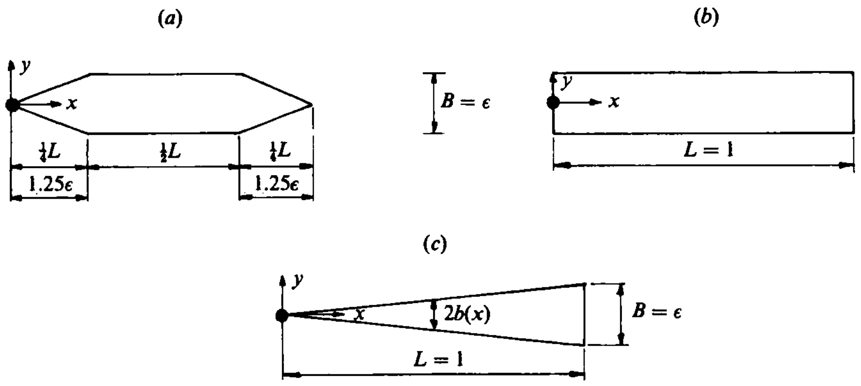

FIGURE 2. Geometries used in the numerical experiments ($\epsilon = 0.20$). (*a*) 'Almost uniform' cross-section. (*b*) Uniform cross-section. (*c*) Non-uniform cross-section.

All numerical experiments have shown an error in $M(x;\alpha)$ consistently smaller than the one in $H(x;\alpha)$. Thus only the error in $H(x;\alpha)$ will be presented and it is defined by the formula

$$e(x) = \frac{H(x;\alpha)|_{\mathrm{E}} - H(x;\alpha)|_{\mathrm{A}}}{H(x;\alpha)|_{\mathrm{E}}} 100\,\%. \tag{9.2}$$

where $H(x;\alpha)|_{\mathrm{E}}$ is the (numerical) 'exact solution' and $H(x;\alpha)|_{\mathrm{A}}$ is the value we obtain from the approximations indicated below:

FK = Froude–Krilov approximation

ST = strip theory

PARAB = parabolic approximation

SBT = slender-body theory, *leading*-order term.

The geometries (*a*), (*b*) in figure 2 have been analysed for two wavelengths ($K_0 L = 1$, long wave; $K_0 B = 1$, short wave) and three incidence angles ($\alpha = 0°; 45°; 90°$). Notice that body (*a*) has an 'almost uniform' cross-section and body (*b*) a uniform cross-section. Body (*c*), with a non-uniform cross-section, will be discussed later in this section. The results obtained are given in tables 1–6, and the last line gives the error of the resultant heave force.

The observed error is certainly under suspicion below the value 0.3 %, the assessed error of the numerical solution of the full linear theory. The error of the leading-order slender-body theory (SBT) must then be taken as being of order 0.3 % which is close to, although smaller, than the predicted one of $\epsilon^3 = 0.8\,\%$. This is certainly due to the fact that diffraction is very weak here, as the error in FK, of order 5 %, shows.

The parabolic approximation, with an average error of order 1.5 %, improves the FK approximation, but it is important to notice that the error in strip theory, of order 10 %, although consistent, is larger than FK. So the apparently more sophisticated ST worsens Froude–Krilov approximation in the long-wave regime.

In the short-wave regime Froude–Krilov approximation has an error of order 50 %, and this indicates the importance of diffraction in this case. Strip theory has an error of order 25 %, close to the estimated value of $\epsilon = 20\,\%$, although the parabolic approximation has an error of order 5 %, roughly one-quarter of the anticipated one. The leading-order slender-body theory has an error around 2 %, half

Section	FK	Body (*a*) ST	SBT	FK	Body (*b*) ST	SBT
0	0	0	0	5.3	6.5	0.38
1	4.2	2.7	0.04	5.7	6.8	0.14
2	4.6	3.4	0.04	6.0	7.0	0.08
3	4.9	5.4	0.004	6.3	7.2	0.06
4	5.1	7.7	0.12	6.6	7.3	0.05
5	5.4	7.8	0.07	6.7	7.4	0.04
6	5.5	7.9	0.05	6.9	7.5	0.04
7	5.6	8.0	0.05	6.9	7.6	0.03
8	5.7	8.0	0.04	7.0	7.6	0.03
9	5.6	8.0	0.05	6.9	7.6	0.03
10	5.5	7.9	0.05	6.9	7.5	0.04
11	5.4	7.8	0.07	6.7	7.4	0.04
12	5.1	7.7	0.12	6.6	7.3	0.05
13	4.9	5.4	0.004	6.3	7.2	0.06
14	4.6	3.4	0.04	6.0	7.0	0.08
15	4.2	2.7	0.04	5.7	6.8	0.14
16	0	0	0	5.3	6.5	0.38
Resultant	5.2 %	6.8 %	0.05 %	6.4 %	7.2 %	0.08 %

TABLE 1. Long waves ($K_0 L = 1$); $\alpha = 90°$. Percentage error (9.2).

Section	FK	Body (*a*) ST	SBT	FK	Body (*b*) ST	SBT
0	0	0	0	4.8	9.7	0.38
1	3.9	1.4	0.05	5.1	9.7	0.14
2	4.3	3.9	0.04	5.5	9.8	0.08
3	4.6	7.3	0.01	5.8	9.9	0.06
4	4.9	10.8	0.11	6.1	10.1	0.04
5	5.1	11.0	0.07	6.4	10.3	0.04
6	5.4	11.2	0.05	6.6	10.4	0.03
7	5.5	11.3	0.04	6.7	10.6	0.03
8	5.6	11.5	0.04	6.8	10.8	0.03
9	5.6	11.7	0.04	6.9	11.0	0.03
10	5.6	11.8	0.05	6.9	11.2	0.03
11	5.5	12.9	0.07	6.9	11.3	0.03
12	5.3	12.0	0.11	6.8	11.4	0.04
13	5.1	8.7	0.01	6.6	11.5	0.06
14	4.8	5.7	0.05	6.4	11.6	0.08
15	4.4	3.6	0.05	6.1	11.6	0.14
16	0	0	0	5.7	11.5	0.38
Resultant	5.1 %	10.0 %	0.04 %	6.2 %	10.7 %	0.08 %

TABLE 2. Long waves ($K_0 L = 1$); $\alpha = 45°$. Percentage error (9.2).

of the predicted value of $\epsilon^2 = 4\,\%$, and we notice that at the ends of a body with blunt ends (body (*b*)) the error is of order 10 %, almost half of the estimated value of $\epsilon = 20\,\%$, see (7.36).

The observed errors follow, as a rule, the theoretical predictions, but in general they are smaller, with one single exception, the error of strip theory in short waves and $\alpha = 45°$ is a little larger than $\epsilon = 20\,\%$. This certainly shows the importance of the longitudinal flow interaction for a body not too slender.

Section	FK	Body (a) ST	SBT	FK	Body (a) ST	SBT
0	0	0	0	4.5	4.7	0.38
1	3.8	49.6	0.045	4.8	3.8	0.14
2	4.1	24.2	0.047	5.2	3.2	0.09
3	4.4	15.6	0.022	5.5	2.8	0.06
4	4.7	11.2	0.112	5.9	2.5	0.05
5	5.0	2.3	0.066	6.1	2.3	0.04
6	5.2	2.0	0.047	6.4	2.1	0.03
7	5.4	1.7	0.037	6.6	1.9	0.02
8	5.5	1.5	0.033	6.7	1.8	0.02
9	5.6	1.4	0.035	6.8	1.7	0.02
10	5.6	1.4	0.042	6.9	1.6	0.02
11	5.4	1.5	0.060	6.9	1.6	0.03
12	5.3	1.6	0.106	6.8	1.6	0.03
13	5.1	17.1	0.017	6.6	1.7	0.05
14	4.8	25.0	0.048	6.4	1.8	0.07
15	4.5	49.6	0.048	6.1	2.0	0.13
16	0	0	0	5.7	2.4	0.37
Resultant	5.0 %	0.2 %	0.038 %	6.0 %	1.5 %	0.07 %

TABLE 3. Long waves ($K_0 L = 1$); $\alpha = 0°$. Percentage error (9.2).

Section	FK	Body (a) ST	SBT	FK	Body (b) ST	SBT
0	0	0	0	24.0	26.0	9.2
1	22.3	11.5	1.2	31.4	19.4	2.6
2	29.0	4.4	1.3	39.7	13.6	1.4
3	37.3	5.7	0.8	47.1	9.8	1.1
4	45.5	12.5	2.7	53.1	9.4	1.1
5	52.7	14.2	1.7	57.6	11.4	1.2
6	58.1	16.8	1.4	60.6	13.7	1.3
7	61.3	18.8	1.3	62.4	15.2	1.4
8	62.4	19.5	1.3	62.9	15.7	1.4
9	61.3	18.8	1.3	62.4	15.2	1.4
10	58.1	16.8	1.4	60.6	13.7	1.3
11	52.7	14.2	1.7	57.6	11.4	1.2
12	45.5	12.5	2.7	53.1	9.4	1.1
13	37.3	5.7	0.8	47.1	9.8	1.1
14	29.0	4.4	1.3	39.7	13.6	1.4
15	22.3	11.5	1.2	31.4	19.4	2.6
16	0	0	0	24.0	26.0	9.2
Resultant	48.6 %	20.2 %	1.2 %	49.5 %	7.7 %	1.8 %

TABLE 4. Short waves ($K_0 B = 1$); $\alpha = 90°$. Percentage error (9.2).

From the theory we obtain also that the leading term of the slender-body theory should have an error of the form $[1+O(\epsilon)]$ when the cross-section is 'non-uniform', as for body (c) in figure 2. It seems, however, difficult to assess this behaviour numerically as the analysis that follows shows.

Section	Body (a) FK	Body (a) ST	Body (a) SBT	Body (b) FK	Body (b) ST	Body (b) SBT
0	0	0	0	17.0	49.1	11.8
1	9.9	9.3	1.1	23.0	45.3	4.5
2	14.9	19.9	1.4	29.1	42.5	2.8
3	21.4	30.8	1.3	34.4	40.1	2.0
4	27.9	41.2	3.3	39.1	37.2	1.4
5	35.9	36.1	1.8	43.6	33.7	0.9
6	42.5	31.3	1.1	48.3	29.5	0.6
7	49.5	27.3	0.7	53.4	25.0	0.3
8	56.9	24.8	0.4	59.0	20.8	0.2
9	63.4	24.7	0.3	65.0	18.1	0.4
10	68.6	26.9	0.5	70.9	17.8	0.6
11	71.6	30.2	1.0	76.5	20.2	0.7
12	71.2	33.0	2.4	80.8	24.2	0.9
13	67.8	32.3	1.8	82.8	28.3	1.2
14	60.4	32.9	1.9	81.0	31.0	1.8
15	51.7	35.8	1.5	74.3	31.1	3.2
16	0	0	0	63.5	27.9	10.2
Resultant	48.2 %	25.5 %	0.7 %	54.9 %	27.8 %	1.8 %

TABLE 5. Short waves ($K_0 B = 1$); $\alpha = 45°$. Percentage error (9.2).

Section	Body (a) FK	Body (a) Parab	Body (a) SBT	Body (b) FK	Body (b) Parab	Body (b) SBT
0	0	0	0	15.6	23.3	12.7
1	9.3	52.3	1.0	21.5	9.4	5.0
2	14.0	27.3	1.7	28.2	6.3	3.3
3	19.8	17.3	2.2	34.5	5.1	2.4
4	25.2	10.0	4.1	39.7	4.2	1.9
5	31.4	4.9	2.5	44.1	3.4	1.5
6	37.6	2.8	1.8	47.8	2.5	1.1
7	44.2	2.8	1.4	51.6	1.6	0.9
8	51.3	4.0	1.2	55.9	1.3	0.6
9	58.9	5.4	0.9	61.4	2.2	0.4
10	66.5	6.7	0.7	68.2	3.7	0.2
11	73.0	8.0	0.5	76.1	5.3	0.2
12	76.9	9.2	1.4	84.8	7.1	0.3
13	78.9	25.1	2.6	92.9	9.0	0.6
14	74.7	29.0	2.8	98.1	11.0	1.2
15	67.4	49.5	2.1	97.1	13.0	2.7
16	0	0	0	87.9	15.3	9.9
Resultant	45.8 %	1.5 %	0.9 %	54.1 %	4.4 %	1.6 %

TABLE 6. Short waves ($K_0 B = 1$); $\alpha = 0°$. Percentage error (9.2).

In fact, from the integral equation (8.7) we obtain, for short waves and $\sin\alpha \sim O(1)$,

$$\tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}I_1}{\mathrm{d}x}(x;\alpha) = -\frac{\mathrm{d}R_S}{\mathrm{d}x}(1+O(\epsilon)),$$

$$\tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}Q_1/\mathrm{d}x}{K_0\sin\alpha} = -\frac{\mathrm{d}R_S}{\mathrm{d}x}(1+O(\epsilon)),$$

Section	FK	ST	SBT
0	0	0	0
1	2.99	2.98	0.09
2	3.66	7.26	0.15
3	4.11	11.05	0.23
4	4.89	13.88	0.29
5	6.92	15.58	0.33
6	10.44	16.13	0.33
7	15.14	15.73	0.30
8	20.82	14.66	0.26
9	27.09	13.53	0.22
10	33.72	13.28	0.26
11	40.34	14.99	0.39
12	46.38	19.02	0.55
13	50.88	24.80	0.72
14	52.48	31.28	1.11
15	49.77	37.12	2.56
16	42.55	40.94	10.37
Resultant	33.14 %	20.82 %	1.02 %

TABLE 7. Error (9.2) for geometry (*c*), figure 2. Short waves ($K_0 B = 1$); $\alpha = 45°$.

where we have used (7.20) and (7.31). It follows, then, that $\partial\Phi^{(1)}/\partial x$ is proportional to $\{\mathrm{d}R_S/\mathrm{d}x;\ \mathrm{d}R_A/\mathrm{d}x\}$, or to $K_0(\mathrm{d}b/\mathrm{d}x)\sin\alpha$. Since the local transverse lengthscale is $b(x)$, we obtain from (4.4)

$$\Phi^{(2)} \sim O\left(2K_0 b^2(x)\cos\alpha K_0 \frac{\mathrm{d}b}{\mathrm{d}x}\sin\alpha\right).$$

If $e(x)$ is the error of the leading-order term we can write ($g(x) = 2b(x)/B$ and $B = \epsilon$)

$$e(x) \approx \tfrac{1}{8}\epsilon(K_0 B)^2 g^2(x)\frac{\mathrm{d}g}{\mathrm{d}x}(x)\sin 2\alpha.$$

For the geometry indicated in figure 2(*c*), $g(x) = x$ and so, when $K_0 B \sim O(1)$,

$$e(x) \approx \tfrac{1}{8}\epsilon x^2$$

or

$$\bar{e} = \int_0^1 e(x)\,\mathrm{d}x \approx \tfrac{1}{24}\epsilon.$$

Such a value of the average error $\bar{e}$ can be numerically distinguished from ϵ^2 only when $\epsilon \ll \frac{1}{24}$, in which case $\bar{e} \ll 0.6\,\%$. This is certainly a very difficult numerical task. For $\epsilon = 0.20$ the error is of order of 0.8 % and the values in table 7 confirm this behaviour.

In the neighbourhood of $x = 0$ the wave is actually very long, since the local transverse lengthscale is $b(x)$ with $K_0 b(x) \ll 1$. This is why the Froude–Krilov approximation worsens when $x \rightarrow L = 1$. The leading-order slender-body theory (SBT) has an average error of order 1 %, close to the one predicted for this geometry, and at the blunt end $x = L$ the error is of order 10 %, again half of the predicted value (equation (7.36)).

On the whole there seems to be a good agreement between the observed errors and the predicted ones. The most important conclusion, however, is the demonstration that the leading-order term has an error factor of the form $[1 + O(\epsilon^2)]$ for a body with an 'almost uniform' cross-section. This result explains why this approximation works so well even for a body with 'blunt ends'.

Appendix. Fourier transform in the class B

We assume first that $P(x) \in B_{\mathrm{E}}[0;L]$ and let

$$\Delta \dot{P}_j = \frac{\mathrm{d}P}{\mathrm{d}x}(x_j^+) - \frac{\mathrm{d}P}{\mathrm{d}x}(x_j^-) \quad (j = 0, 1, \ldots, e).$$

From the properties of the class $B_{\mathrm{E}}[0;L]$ and a routine analysis of Fourier transforms we can easily check that

$$P^*(K) = -\frac{1}{K^2} \sum_{j=0}^{e} \Delta \dot{P}_j \, \mathrm{e}^{\mathrm{i}Kx_j} + \frac{F(K)}{K^3}, \quad F(K) \sim O(P). \tag{A 1}$$

If we now define

$$\overline{P}(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} P^*(K) \, \mathrm{e}^{-\mathrm{i}Kx} \mathrm{d}K \quad (\lambda \gg 1), \tag{A 2}$$

we obtain

$$P(x) = \overline{P}(x) + E(x), \tag{A 3}$$

where

$$\left.\begin{aligned} E(x) &= P_{\mathrm{R}}(x) + P_{\mathrm{L}}(x) \\ P_{\mathrm{R}}(x) &= \frac{1}{2\pi} \int_{\lambda}^{\infty} P^*(K) \, \mathrm{e}^{-\mathrm{i}Kx} \, \mathrm{d}K, \\ P_{\mathrm{L}}(x) &= \frac{1}{2\pi} \int_{-\infty}^{-\lambda} P^*(K) \, \mathrm{e}^{-\mathrm{i}Kx} \, \mathrm{d}K. \end{aligned}\right\} \tag{A 4}$$

Using (A 1) in (A 4) we get

$$\left.\begin{aligned} P_{\mathrm{R}}(x) &= \left[-\frac{1}{2\pi} \sum_{j=0}^{e} \Delta \dot{P}_j \int_{\lambda}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}K(x-x_j)}}{K^2} \mathrm{d}K \right] \left(1 + O\left(\frac{1}{\lambda^2}\right) \right), \\ P_{\mathrm{L}}(x) &= \left[-\frac{1}{2\pi} \sum_{j=0}^{e} \Delta \dot{P}_j \int_{-\infty}^{-\lambda} \frac{\mathrm{e}^{-\mathrm{i}K(x-x_j)}}{K^2} \mathrm{d}K \right] \left(1 + O\left(\frac{1}{\lambda^2}\right) \right). \end{aligned}\right\} \tag{A 5}$$

If we define

$$D_j(x) = -\frac{1}{2\pi} \left[\int_{-\infty}^{-\lambda} + \int_{\lambda}^{\infty} \frac{\mathrm{e}^{\mathrm{i}K(x-x_j)}}{K^2} \mathrm{d}K \right], \tag{A 6}$$

then from (A 3), (A 4), (A 5) we obtain

$$P(x) \doteq \left[\overline{P}(x) + \sum_{j=0}^{e} \Delta \dot{P}_j D_j(x) \right] \left(1 + O\left(\frac{1}{\lambda^2}\right) \right), \quad P(x) \in B_{\mathrm{E}}[0;L]. \tag{A 7}$$

If $P(x) \in B[0;L]$ we can write $(0 \leqslant x \leqslant L)$

$$P(x) = P(0) + [P(L) - P(0)] \frac{x}{L} + P_{\mathrm{E}}(x), \quad P_{\mathrm{E}}(x) \in B_{\mathrm{E}}[0;L],$$

$$P(0) = \lim_{x \to 0^+} P(x), \quad P(L) = \lim_{x \to L^-} P(x).$$

Then
$$P^*(K) = \mathrm{i}\frac{P(0)}{K} - \mathrm{i}\frac{P(L)}{K} \mathrm{e}^{\mathrm{i}KL} - \frac{1}{K^2} \frac{P(L) - P(0)}{L} (1 - \mathrm{e}^{\mathrm{i}KL}) + P_{\mathrm{E}}^*(K),$$

and so

$$P(x) \doteq \left[\overline{P}(x) + \sum_{j=0}^{e} \Delta \dot{P}_j D_j(x) + P(0) \frac{\mathrm{d}D_0}{\mathrm{d}x}(x) + P(L) \frac{\mathrm{d}D_e}{\mathrm{d}x}(x) \right] \left(1 + O\left(\frac{1}{\lambda^2}\right) \right),$$

$$P(x) \in B[0;L]. \tag{A 8}$$

From (A 6) we get

$$D_j(x) = -\frac{1}{2\pi\lambda} D(\lambda|x-x_j)|), \tag{A 9}$$

where

$$\left.\begin{aligned} D(z) &= z\left[\int_{-\infty}^{-z} + \int_{z}^{\infty} \frac{e^{-it}}{t^2}\,dt\right] = -2\int_{z}^{\infty} \mathrm{si}(\xi)\,d\xi, \\ \mathrm{si}(\xi) &= -\int_{\xi}^{\infty} \frac{\sin t}{t}\,dt. \end{aligned}\right\} \tag{A 10}$$

Expression (A 10) can be obtained, after some algebra, by integrating e^{-it}/t^2 in the complex plane.

From the properties of $D_j(x)$ we obtain

$$\left\{D_j(x); \frac{1}{\lambda^n}\frac{d^n D_z}{dx^n}(x)\right\} \sim \begin{cases} O(1/\lambda) & \text{if } |x-x_j| \leqslant O(1/\lambda), \\ O(1/\lambda^2) & \text{if } |x-x_j| \sim O(1), \end{cases}$$

or, with the definition (6.8),

$$\left\{D_j(x); \frac{1}{\lambda^n}\frac{d^n D_j}{dx^n}(x)\right\} \sim O(1/\lambda^2). \tag{A 11}$$

The function $\overline{P}(x)$ is certainly infinitely differentiable, and from (A 7), (A 8), (A 11) we obtain

$$\left.\begin{aligned} P(x) &= \overline{P}(x)[1+O(1/\lambda^2)] \\ \frac{1}{\lambda^n}\frac{d^n\overline{P}}{dx^n} &\sim O(1/\lambda^2) \quad (n = 2, 3, \ldots), \end{aligned}\right\} \quad (P(x) \in B_{\mathrm{E}}[0; L]), \tag{A 12}$$

or

$$\left.\begin{aligned} P(x) &= \overline{P}(x)[1+O(1/\lambda)] \\ \frac{1}{\lambda^n}\frac{d^n\overline{P}}{dx^n} &\sim O(1/\lambda) \quad (n = 1, 2, \ldots). \end{aligned}\right\} \quad (P(x) \in B[0, L]), \tag{A 13}$$

These relations demonstrate (6.9), (6.11). We want next to show that, for short waves and an arbitrary $P(x) \in B[0; L]$ we have, for $0^+ \leqslant x \leqslant L^-$,

$$\left.\begin{aligned} \frac{dI_P}{dx}(x;\alpha) &= \frac{dP/dx}{K_0 \sin\alpha}(1+O(\epsilon)) \quad (\sin\alpha \sim O(1)), \\ \frac{dI_P}{dx}(x;\alpha) &= \left[\frac{1-i}{(\pi K_0)^{\frac{1}{2}}}\int_0^x d\xi \frac{dP/d\xi}{(x-\xi)^{\frac{1}{2}}}\right][(1+O(\epsilon)) \quad (\alpha = 0), \end{aligned}\right\} \tag{A 14}$$

where $I_P(x;\alpha) = \mathscr{L}(P)$; see (6.14). In order to do so we first introduce the function

$$\dot{P}(x) = \begin{cases} = \dfrac{dP}{dx} & (0^+ \leqslant x \leqslant L^-), \\ = 0 & (x \leqslant 0^- \quad \text{or } x \leqslant L^+). \end{cases} \tag{A 15}$$

The function $\dot{P}(x)$ avoids the eventual Dirac δ-function of dP/dx in $x = 0$ or $x = L$. We define next

$$\dot{I}_P(x;\alpha) = \mathscr{L}(\dot{P}) = \frac{1}{2}\int_{0^+}^{L^-} d\xi \frac{dP}{d\xi}(\xi)\, H_0^{(1)}(K_0|x-\xi|)\, e^{-iK_0(x-\xi)\cos\alpha}. \tag{A 16}$$

The function $\dot{P}(x)$ is in a class $\hat{B}[0, L]$ with the same sort of singularities as $B[0; L]$.

It follows then that

$$\left.\begin{aligned} \mathring{I}_P(x;\alpha) &= \frac{\mathrm{d}P/\mathrm{d}x}{K_0 \sin\alpha}(1+O(\epsilon)) \quad (\sin\alpha \sim O(1)), \\ \mathring{I}_P(x;\alpha) &= \left[\frac{1-\mathrm{i}}{(\pi K_0)^{\frac{1}{2}}}\int_0^x \mathrm{d}\xi \frac{\mathrm{d}P/\mathrm{d}\xi}{(x-\xi)^{\frac{1}{2}}}\right](1+O(\epsilon)) \quad (\alpha = 0) \end{aligned}\right\} \tag{A 17}$$

when $K_0 B \sim O(1)$ (short waves).

The function $\mathring{I}_P(x;\alpha)$ is closely related to $(\mathrm{d}I_P/\mathrm{d}x)(x;\alpha)$. In fact, since

$$I_P(x;\alpha) = \int_{0^+}^{L^-} P(\xi)\, g(x-\xi)\,\mathrm{d}\xi,$$

$$g(x) = \tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}K_0 x\cos\alpha} H_0^{(1)}(K_0|x|),$$

then, differentiating the above equality with respect to x and integrating by parts, we obtain

$$\frac{\mathrm{d}I_P}{\mathrm{d}x}(x;\alpha) = \mathring{I}_P(x;\alpha) + \Delta P(x;\alpha), \tag{A 18}$$

where

$$\left.\begin{aligned} \Delta P(x;\alpha) &= \tfrac{1}{2}[P(0)\,\mathrm{e}^{-\mathrm{i}K_0 x\cos\alpha} H_0^{(1)}(K_0|x|) - P(L)\,\mathrm{e}^{\mathrm{i}K_0\bar{x}\cos\alpha} H_0^{(1)}(K_0\bar{x})], \\ \bar{x} &= x - L. \end{aligned}\right\} \tag{A 19}$$

If $P(x) \in B_{\mathrm{E}}[0;L]$ then $P(0) = P(L) = 0$ and (A 14) follows at once from (A 17), (A 18). In the metric (6.8) the function $\Delta P(x;\alpha)$ is associated with the linear functional

$$\Delta P(\Psi) = \int_{-\infty}^{\infty} \Delta P(x;\alpha)\,\Psi(x)\,\mathrm{d}x \tag{A 20}$$

and we shall show next that for *any* $P(x) \in B[0;L]$ we have, for *short waves and* $\sin\alpha \sim O(1)$,

$$\Delta P(\Psi) = \left[\frac{P(0)}{K_0 \sin\alpha}\Psi(0) - \frac{P(L)}{K_0 \sin\alpha}\Psi(L)\right](1+O(\epsilon)), \tag{A 21a}$$

or

$$\Delta P(x) = \left[\frac{P(0)}{K_0 \sin\alpha}\delta(x) - \frac{P(L)}{K_0 \sin\alpha}\delta(x-L)\right](1+O(\epsilon)), \tag{A 21b}$$

and for *short waves and* $\alpha = 0$

$$\Delta P(x) \sim 0(\epsilon). \tag{A 21c}$$

Obviously (A 21), together with (A 17), demonstrate (A 14) in the interval $0^+ \leqslant x \leqslant L^-$. To demonstrate (A 21) we observe that

$$\Delta P(\Psi) = P(0)\,I_1(\Psi) - P(L)\,I_2(\Psi), \tag{A 22}$$

where

$$\left.\begin{aligned} I_1(\Psi) &= \frac{1}{2}\int_{-\infty}^{\infty} \Psi(x)\,\mathrm{e}^{-\mathrm{i}K_0 x\cos\alpha} H_0^{(1)}(K_0|x|)\,\mathrm{d}x, \\ I_2(\Psi) &= \frac{1}{2}\int_{-\infty}^{\infty} \Gamma(\bar{x})\,{}^{\mathrm{i}K_0\bar{x}\cos\alpha} H_0^{(1)}(K_0|\bar{x}|)\,\mathrm{d}\bar{x}, \\ \Gamma(\bar{x}) &= \Psi(x) = \Psi(\bar{x}+L);\ \bar{x} = x - L. \end{aligned}\right\} \tag{A 23}$$

Obviously

$$\left.\begin{aligned} I_1(\Psi) &= \Psi_1(x;\alpha)\,|_{x=0}, \\ I_2(\Psi) &= \Gamma_1(\bar{x};\alpha)\,|_{\bar{x}=0}, \end{aligned}\right\} \tag{A 24}$$

where

$$\left.\begin{aligned}\Psi_1(x;\alpha) &= \frac{1}{2}\int_{-\infty}^{\infty} \Psi(\xi)\,\mathrm{e}^{\mathrm{i}K_0(x-\xi)\cos\alpha}H_0^{(1)}(K_0|x-\xi|)\,\mathrm{d}\xi = \mathscr{L}(\overline{\Psi}),\\ \Gamma_1(x;\alpha) &= \frac{1}{2}\int_{-\infty}^{\infty} \Gamma(\xi)\,\mathrm{e}^{-\mathrm{i}K_0(x-\xi)\cos\alpha}H_0^{(1)}(K_0|x-\xi|)\,\mathrm{d}\xi = \mathscr{L}(\Gamma),\\ \overline{\Psi}(x) &= \Psi(-x).\end{aligned}\right\} \tag{A 25}$$

Since both $\{\overline{\Psi}(x);\,\Gamma(x)\}$ are 'good functions' (see (A 25), (A 23), (A 20)), they are smoother than the functions in class $B_{\mathrm{E}}[0;L]$ and so, using the results derived in §§7.2 and 7.3, we obtain

(a) short waves; $\sin\alpha \sim O(1)$,

$$\left.\begin{aligned}\Psi_1(x;\alpha) &= \mathscr{L}(\overline{\Psi}) \doteq \frac{\Psi(-x)}{K_0\sin\alpha}(1+O(\epsilon)),\\ \Gamma_1(x;\alpha) &= \mathscr{L}(\Gamma) \doteq \frac{\Gamma(\overline{x})}{K_0\sin\alpha}(1+O(\epsilon)),\end{aligned}\right\} \tag{A 26}$$

(b) short waves; $\alpha = 0$,

$$\left.\begin{aligned}\Psi_1(x;0) &= \mathscr{L}(\overline{\Psi}) \doteq \left[\frac{1-\mathrm{i}}{(\pi K_0)^{\frac{1}{2}}}\int_0^{x}\mathrm{d}\xi\frac{\Psi(-\xi)}{(x-\xi)^{\frac{1}{2}}}\right](1+O(\epsilon)),\\ \Gamma_1(x;0) &= \mathscr{L}(\Gamma) \doteq \left[\frac{1-\mathrm{i}}{(\pi K_0)^{\frac{1}{2}}}\int_0^{\overline{x}}\mathrm{d}\xi\frac{\Gamma(\xi)}{(\overline{x}-\xi)^{\frac{1}{2}}}\right](1+O(\epsilon)).\end{aligned}\right\} \tag{A 27}$$

We emphasize that the approximations (A 26) and (A 27) are uniform in x since, in the class $B_{\mathrm{E}}[0;\,L]$,

$$P(x) \doteq \overline{P}(x)\,(1+O(\epsilon)),$$

see (A7). This is why we have used the symbol $\doteq$, and from the uniformity we obtain

(a) short waves; $\sin\alpha \sim O(1)$

$$\left.\begin{aligned}\Psi_1(0;\alpha) &= \frac{\Psi(0)}{K_0\sin\alpha}(1+O(\epsilon)),\\ \Gamma_1(0;\alpha) &= \frac{\Gamma(0)}{K_0\sin\alpha}(1+O(\epsilon)) = \frac{\Psi(L)}{K_0\sin\alpha}(1+O(\epsilon)),\end{aligned}\right\} \tag{A 28}$$

(b) short waves; $\alpha = 0$

$$\Psi_1(0;0) \sim O(\epsilon), \quad \Gamma_1(0;0) \sim O(\epsilon). \tag{A 29}$$

Using (A 28), (A 29) in (A 24) and (A 22), we can easily check (A 21). In this way we have demonstrated (A 14) in the whole class $B[0;L]$.

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